MELNIKOV TRANSFORMS, BERNOULLI BUNDLES, AND ALMOST PERIODIC PERTURBATIONS

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ABSTRACT. In this paper we study nonlinear time-varying perturbations of an autonomous vector field in the plane R^2. We assume that the unperturbed equation, i.e. the given vector field has a homoclinic orbit and we present a generalization of the Melnikov method which allows us to show that the perturbed equation has a transversal homoclinic trajectory. The key to our generalization is the concept of the Melnikov transform, which is a linear transformation on the space of perturbation functions.

The appropriate dynamical setting for studying these perturbation is the concept of a skew product flow. The concept of transversality we require is best understood in this context. Under conditions whereby the perturbed equation admits a transversal homoclinic trajectory, we also study the dynamics of the perturbed vector field in the vicinity of this trajectory in the skew product flow. We show the dynamics near this trajectory can have the exotic behavior of the Bernoulli shift. The exact description of this dynamical phenomenon is in terms of a flow on a fiber bundle, which we call, the Bernoulli bundle. We allow all perturbations which are bounded and uniformly continuous in time. Thus our theory includes the classical periodic perturbations studied by Melnikov, quasi periodic and almost periodic perturbations, as well as toroidal perturbations which are close to quasi periodic perturbations.

I. INTRODUCTION

In a seminal paper Smale (1963) introduced a very interesting geometric construction for showing that certain diffeomorphisms have infinitely many periodic points. The basic idea in this construction was to show that the existence of a certain "horseshoe" region implied that a full Bernoulli shift could be imbedded into the dynamics of the diffeomorphism. This horseshoe construction has since become an important technique in the study of dynamical systems, see for example Devaney (1986). While the horseshoe construction has many nice features, it is not without shortcomings. For example, in some problems the

location of a horseshoe region is not always obvious, and sometimes requires extensive argumentation simply to show its existence.

More recently Palmer (1984) used the Lyapunov-Perron method (of integral equations) and the theory of exponential dichotomies to develop an analytical approach, which when combined with the Melnikov method, has several very attractive features. It avoids completely the need for constructing a horseshoe region while at the same time gives a transparent and elementary proof of the imbedding of the Bernoulli shift flow. Also, as we shall see, it is the appropriate point of view for studying almost periodic perturbations of systems containing homoclinic orbits.

By extending Palmer's ideas, we present here a framework for generalizing the Bernoulli imbedding of Smale (1963) and the method of detecting transversal homoclinic orbits of Melnikov (1963) to almost periodic systems of differential equations. The key to this generalization is to use the Miller (1965) and Sell (1967) construction of a skew product flow over the hull of an almost periodic function. In this way the nonautonomous equations define a dynamical system whose structure reflects the geometry and spectra of the almost periodic forcing term. The main results of this paper were announced in Meyer and Sell (1986).

Within the context of this skew product dynamical system we provide the appropriate generalizations of the following concepts: hyperbolic invariant sets, the stable and unstable manifolds, transversal homoclinic orbits, and shift automorphisms on a symbol space. This last object we call a Bernoulli bundle because it is a fiber bundle with fiber maps which are Bernoulli automorphisms. We then proceed to prove generalizations of (i) the Melnikov theorem for detecting homoclinic orbits, (ii) the shadowing lemma and (iii) Smale's theorem on the existence of a Bernoulli shift in the flow. One of the key ingredients for these generalizations is the Melnikov transform, which we introduce in this paper. In the context of almost periodic functions the Melnikov transform is a flow preserving mapping of the hull of one almost periodic function onto the hull of another. As we shall see, certain properties of the Melnikov transform will determine respective dynamical properties of the perturbed differential equation.

We first consider a basic model equation of the form

$$(1.1) \qquad\qquad \dot{x} = F(x) + \varepsilon f(t, x)$$

where $x \in R^2, t \in R, \cdot = d/dt, F: R^2 \to R^2$ is smooth and $f \in \mathscr{A}\mathscr{P}$. Here $\mathscr{A}\mathscr{P}$ is the collection of all functions $g: R \times R^2 \to R^2$ where g is almost periodic in t, uniformly for x, in compact subsets of R^2. The space $\mathscr{A}\mathscr{P}$ has the compact open topology, i.e. the topology of uniform convergence on compact subsets of $R \times R^2$. We assume that the autonomous unperturbed system, when $\varepsilon = 0$, has a nondegenerate saddle point v_0 and a homoclinic orbit (saddle connection) u^0 as pictured in Figure 1. Instead of just considering one function, we take $f \in H \subset \mathscr{A}\mathscr{P}$ where $H = H(g)$ is the hull of an almost periodic function g. Recall that if $g: R \times R^2 \to R^2$ is almost periodic and
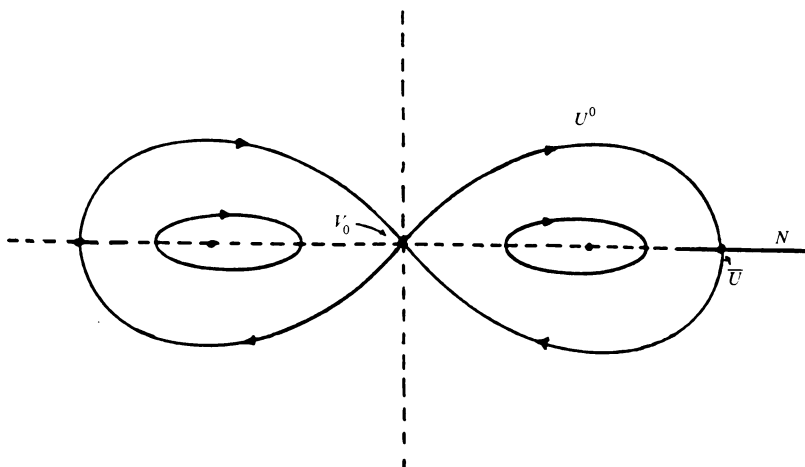
FIGURE 1. Duffing's flow: The unperturbed phase por-
trait

$\tau \in R$, the τ-translate of g is $g_\tau(t, x) = g(t + \tau, x)$ and $H(g)$ is the closure, in the compact open topology, of the set of all translates of g.

The translational flow on $\mathscr{A}\mathscr{P}$ is defined by

$$\pi: R \times \mathscr{A}\mathscr{P} \to \mathscr{A}\mathscr{P}: (\tau, f) \to f_\tau.$$

For any $g \in \mathscr{A}\mathscr{P}$, the hull $H(g)$ is a compact invariant minimal set. When g is quasi periodic H turns out to be homeomorphic to a torus and this flow is equivalent to the standard irrational twist flow. When g is limit periodic, H is homeomorphic to a standard solenoid and the flow is equivalent to the suspension of the classical adding machine map. §2 contains background material on almost periodic functions, the flow on their hull, cross sections to this flow and the general construction of the Bernoulli bundles. Two prototypical examples of a quasi periodic and a limit periodic function will be introduced in §2 and used to illustrate our theory throughout this paper.

For ε fixed we let $\phi(t, x_0, f)$ denote the solution of (1.1) which goes through x_0 at epoch $t = 0$ and assume that ϕ is globally defined. Then the skew product flow defined by (1.1) is the flow

$$\Pi: R \times (R^2 \times H) \to R^2 \times H: (t, (x_0, f)) \to (\phi(t, x_0, f), f_t).$$

Note the action on the second factor H is just the translational flow π, which we refer to as the base flow. Properties of this skew product flow are discussed in a greater detail in §2, however a few observations are useful at this point.

For $\varepsilon = 0$, Π decouples into a product flow and so the saddle point v_0 and the product of its stable and unstable manifolds with H become important invariant sets for Π. Classical theorems on almost periodic differential equations establish the existence of similar invariant sets for Π when $\varepsilon \neq 0$ but small. Of course these invariant sets for $\varepsilon \neq 0$ need not be product spaces,

but they do reflect the dynamics for $\varepsilon = 0$, see Sacker and Sell (1977) and Sell (1978). For example, the saddle point $\{v_0\} \times H$ becomes an invariant set for the skew product flow which is a 1-fold covering of H and is flow equivalent to the translation flow on H when $0 < |\varepsilon| \ll 1$. These concepts and the generalization of transversal homoclinic orbit are given in §3. Interestingly, the Melnikov transform finds its natural domain of definition on the hull H, and its zero set oftentimes defines a global cross section for the flow π on H. Thus the Melnikov transform defines a cross section which in turn determines a time (the first return time), or a clock, which we use to describe the dynamics of the Bernoulli bundles. This clock is intimately connected with the spectrum of the almost periodic function g and replaces the period in periodic systems. The main result of §3 is that if the Melnikov transform has a simple zero set then there exists a normally hyperbolic homoclinic bundle which, together with a suitable Shadowing Lemma, will beget the Bernoulli bundle.

§4 contains the generalization of hyperbolic invariant set for skew product flows and a simple proof of a generalization of the shadowing lemma based on the functional analytic proof given in Meyer and Sell (1987). The shadowing lemma is the main tool we use to prove the existence of the Bernoulli bundle invariant set. For this purpose we follow Palmer (1984), who used the shadowing lemma to establish the existence of an invariant set in periodic systems which has a cross section map equivalent to the Bernoulli shift automorphism on a Cantor set. This idea avoids the problem of generalizing the geometric construction of the horseshoe region of Smale (1963). If the Melnikov transform has a simple zero set, then it is shown that Π has a hyperbolic invariant set, which is simply the closure of this homoclinic set. The shadowing lemma then establishes the existence of orbits which are close to the homoclinic orbit for long periods but can take arbitrary small jumps from time to time. These theorems along with some examples are given in §5.

While our primary interest is studying equation (1.1) when the perturbation term $f(t, x)$ is almost periodic in t, uniformly for x in compact subsets of R^2, the theory we present here is has broader applications. By using the theory of skew product flows as developed in Sell (1967) and Sacker and Sell (1977) one can handle perturbations where $f(t, x)$ need not be almost periodic in t. For example, if $f(t, x)$ is uniformly continuous on sets of the form $R \times K$, where K is compact in R^2, then the hull is still a compact invariant set. An illustration of our theorems is given in §6 where we study the perturbed Duffing equation with negative linear stiffness

$$
\begin{aligned}
(1.2) \qquad & u'' = u - u^3 + \varepsilon_1 f(u, u', \theta, \varepsilon_1, \varepsilon_2), \\
& \theta' = \omega + \varepsilon_2 h(\theta) + \varepsilon_1 g(u, u', \theta, \varepsilon_1, \varepsilon_2),
\end{aligned}
$$

where $\theta = (\theta_1, \theta_2) \in T^2$, the 2-torus, and $\omega = (\omega_1, \omega_2)$ is a constant vector. If $g \equiv 0$ and $h \equiv 0$, then (1.2) is equivalent to an almost periodic perturbation

$$
u'' = u - u^3 + \varepsilon_1 f(u, u', \omega_1 t, \omega_2 t, \varepsilon_1, \varepsilon_2)
$$

of Duffing's equation. However for generic choices for g and h this perturbation is not almost periodic. What we show is that there is a very large class of perturbations for which (1.2) generates a Bernoulli bundle flow (along with the associated chaotic dynamics). For example, if

$$f(u, u', \theta, 0) = a_1 \cos \theta_1 + a_2 \cos \theta_2$$

where a_1, a_2 are constants, then for a large open set in the parameter space $\{(a_1, a_2, \omega_1, \omega_2)\}$ (see §III.5) and for all g and h, (1.2) has a Bernoulli bundle flow for $0 < |\varepsilon_i| < \varepsilon_0$, $i = 1, 2$, where ε_0 is positive and depends on the size of f, g and h.

While we shall not do it here, one can also study perturbations $f(t, x)$ of (1.1) which are discontinuous in t, by using the generalizations as developed in Miller and Sell (1970), Artstein (1977) and Schwabik (1985). The key to all these considerations is the skew product dynamics which we present in §3.

Homoclinic orbits and their implications for autonomous and periodic systems have been investigated since their introduction by Poincaré and therefore have a vast literature. The classics in the subject are Poincaré (1892), Birkhoff (1932), Cartwright and Littlewood (1945), Melnikov (1963), and Smale (1963). Our work uses many of the ideas of Palmer (1984). The reader is referred to Chow and Hale (1982) and Guckenheimer and Holmes (1983) for a detailed discussion of and historical remarks on the autonomous and periodic literature.

Recently some related work has appeared on almost periodic systems. In Wiggens (1986b) the dynamical behavior of the Bernoulli bundle is described in the quasi periodic case. However, Wiggins (1968a) does not develop the connection with the Melnikov transform which we present here. Scheurle (1986) considers a system of almost periodic equations similar to ours, but considers only one equation and not a whole class of equations based on the hull of an almost periodic function. He uses the theory of exponential dichotomies, as extended by Palmer (1984), to find particular solutions which have a random structure. By not looking at the skew product flow defined by the equation he loses compactness and thus the uniformity needed to see the whole picture.

In another direction Ercolani, Forest and McLaughlin (1987) have shown that chaotic behavior can develop when one introduces arbitrarily small periodic forcing (with dissipation) to the sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0.$$

We believe that a straightforward extension of the theory we develop here can be used to explain this behavior. This problem is addressed in a sequel to this paper, Meyer and Sell (1988).

We express our appreciation to Clark Robinson who suggested the use of subshifts of finite type to describe our results. We also express our appreciation to the referee who pointed out some inaccuracies in an earlier version of this paper and made some other very helpful suggestions.

II. The hull, cross sections and Bernoulli bundles

Throughout this paper *almost periodic* will be in the sense of the founder of the subject, Bohr (1925ab, 1926, 1959); Besicovitch (1932) and others say that f is uniformly almost periodic. Our function f will depend on parameters, $x \in R^n$, and so the uniformity we speak of is with respect to these parameters. In this section we review some basic definitions and theorems so that we can develop the definition of the complicated flow on the spaces we call Bernoulli bundles. Basic references on almost periodic functions are Besicovitch (1932), Favard (1933), Bohr (1959), Corduneanu (1968), Sell (1971), Fink (1974), and Levitan and Zhikov (1982).

II.1 Almost periodic functions. A set $T \subset R = R^1$ is *relatively dense* if there is an $L > 0$ such that any interval in R of length L contains a point of T. Let $f: R \times R^n \to R^n$ (or $R \times C^n \to C^n$) be continuous and let K be a compact set in R^n (or C^n). An (ε, K)-*period* for f is a number τ such that $\|f(t + \tau, x) - f(t, x)\| < \varepsilon$ for all $(t, x) \in R \times K$. f is *almost periodic*, uniformly for x in compact sets, if it is bounded and for every $\varepsilon > 0$ and every compact set K in R^n (or C^n), the set of (ε, K)-periodic for f is relatively dense.

The spectral theory of an almost periodic function is based on the fact that the mean value

$$M_\omega\{f\}(x) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(s, x) e^{-i\omega s} \, ds$$

exists, and for only a countable number of real numbers ω does one have $M_\omega\{f\} \neq 0$. The set $\{\omega: M_\omega\{f\} \neq 0\}$ is called the set of *exponents* or *frequencies* of f. We write

$$(2.1) \qquad\qquad f(t, x) \sim \sum_{-\infty}^{\infty} a_k(x) e^{i w_k t}$$

where $a_k(x) = M_{\omega_k}\{f\}(x)$. This series is called the Fourier series of f. In general, the series given above does not converge to f except in the mean square sense, i.e.,

$$M\left\{\|f(t, x) - p_N(t, x)\|^2\right\} \to 0$$

where $p_N(t, x) = \sum_{|k| \leq N} a_k(x) e^{i\omega_k t}$. Consider the real numbers R as a vector space over the rational numbers Q. The smallest subspace $S \subset R$ which contains the set of exponents of f is called the *modulus* of f.

In the case when S is one dimensional, i.e. $\omega_k = r_k \alpha$, where α is a real number and $r_k \in Q$, the function f is said to be *limit periodic*, because in this case the partial sums p_N of the Fourier series given above are periodic. Also, it can be shown that f is limit periodic if and only if it is the uniform limit of periodic functions (not necessarily the partial sums p_N). The example we use

throughout this paper is

$$\ell(t) = \sum_{0}^{\infty} a_k e^{i2\pi(t/2^k)}$$

(or its real part) where the a_k are constants chosen so that the series converges rapidly. The Nth partial sum of l is 2^N-periodic.

If the modulus S is a finite module over the integers, i.e., there exists a finite set $\lambda_1, \ldots, \lambda_s \in R$ such that $\omega_k = \sum a_{ki}\lambda_i$ with a_{ki} integers, then f is said to be *quasi periodic*. Equivalently, f is quasi periodic if and only if there is a continuous function $F(\theta_1, \ldots, \theta_s)$, that is periodic in each argument, and real numbers $\lambda_1, \ldots, \lambda_s$ such that $f(t) = F(\lambda_1 t, \ldots, \lambda_s t)$. Our standard example of a quasi periodic function will be

$$q(t) = a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}$$

(or its real part) where the ratio ω_1/ω_2 is irrational. Let $\mathscr{C} = \mathscr{C}(R \times R^n, R^n)$ (or $\mathscr{C}(R \times C^n, C^n)$) be the space of continuous functions from R to R^n (or C^n) with the topology of uniform convergence on compact sets (the compact open topology). Translations define a flow on $\mathscr{C}$ as follows

$$\pi: R \times \mathscr{C} \to \mathscr{C}: (\tau, f) \to f_\tau$$

where $f_\tau(t, x) = f(t + \tau, x)$. For any $f \in C$ the orbit closure of f is called the hull of f and is denoted by $H(f)$. That is $H(f) = \text{Cl}\{g_\tau: \tau \in R\}$, where the closure is taken in the compact open topology. Clearly $H(f)$ is invariant, but the hull of an almost periodic function is more. The following result can be found in Nemytskii and Stepanov (1960).

Theorem. *If f is almost periodic, then $H(f)$ is a compact minimal set. Furthermore, each element of $H(f)$ is almost periodic, the restriction $\pi|_{H(f)}$ is equicontinuous, and $H(f)$ can be given a compact, connected, Abelian group structure.*

The group structure $+$ is defined as follows: First define $f_\tau + f_\sigma = f_{\tau+\sigma}$. Then if $f_{\tau_n} \to a$ and $f_{s_n} \to b$ select a subsequence, if necessary, and define $a + b = \lim f_{t_n+s_n}$. It is not difficult to show that this is a well-defined Abelian group operation. The mapping $\tau \to f_\tau$ is a homomorphism of R onto a dense subgroup of $H(f)$ and so $H(f)$ is a compactification of R. One can also use the space $\mathscr{AP}$ of almost periodic functions with supremum norm. The above results hold in this space as well.

If f is almost periodic with a Fourier series $f \sim \sum a_k e^{i\omega_k t}$ then the translate f_τ has the Fourier series $f_\tau \sim \sum a_k e^{i\omega_k(t+\tau)}$. For $g \in H(f)$ we obtain the Fourier series as follows: Let $f_{\tau_n} \to g$. By using the Cantor diagonalization procedure to select a subsequence, if necessary, we can assume that

$$\tau_n \to \phi_k \mod 2\pi/\omega_k \quad \text{for all } k, \text{ as } n \to \infty.$$

Then the Fourier coefficients of f_{t_n} converge to the Fourier coefficients of g or

$$(2.2) \qquad g \sim \sum a_k e^{i\omega_k(t+\phi_k)} .$$

Thus if $g \in H(f)$ the phases ϕ_k are defined mod $2\pi/\omega_k$ such that (2.2) holds. As shown in the next paragraph, the converse is not true, i.e., one cannot choose ϕ_k mod $2\pi/\omega_k$ arbitrarily and find a function $g \in H(f)$ such that (2.2) holds.

Let $p(t) = e^{it} + e^{i2t}$ so p is 2π periodic. It is easy to see that the hull of p is precisely

$$H(p) = \left\{ e^{i(t+\alpha)} + e^{i2(t+\alpha)} : 0 \le \alpha < 2\pi \right\},$$

which does not contain $e^{it} + e^{i2(t+1)}$ for example. It is easy to see that the map $\alpha \to e^{i(t+\alpha)} + e^{i2(t+\alpha)}$ is a topological homeomorphism and a continuous group isomorphism of $S = R/(2\pi Z)$ onto $H(p)$. In general the hull of a periodic function is homeomorphic to a circle.

Let us now introduce the two prototypical examples which we will use to illustrate various aspects of our theory. The first is the quasi periodic function

$$(QP) \qquad q(t) = a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}$$

where a_1, a_2 are nonzero. We will be generally interested in the case where ω_1/ω_2 is irrational. The second example is the limit periodic function

$$(LP) \qquad \ell(t) = \sum_{k=0}^{\infty} a_k e^{i2\pi(t/2^k)}$$

where the a_k's are nonzero and are chosen so that the series in (LP) converges uniformly and absolutely.

For (QP) assume that ω_1/ω_2 is irrational. By Kronecker's theorem,[1] for any pair of real numbers α_1 and α_2 and any $\varepsilon > 0$ there are integers N_1 and N_2 and a real number τ such that $|\tau\omega_i - 2\pi N_i - \alpha_i| < \varepsilon$. Consequently the translate q_τ

$$\begin{aligned} q_\tau(t) &= a_1 e^{iw_1(t+\tau)} + a_2 e^{iw_2(t+\tau)} \\ &= a_1 e^{i(w_1 t + \alpha_1 + \xi_1)} + a_2 e^{i(w_2 t + \alpha_2 + \xi_2)} \end{aligned}$$

where $|\xi_i| < \varepsilon$. By taking $\varepsilon = 1/n$ and finding the corresponding $\tau = \tau_n$, one obtains

$$q_{\tau_n}(t) \to a_1 e^{i(\omega_1 t + \alpha_1)} + a_2 e^{i(\omega_2 t + \alpha_2)}$$

and so

$$H(q) = \{ a_1 e^{i(\omega_1 t + \alpha_1)} + a_2 e^{i(\omega_2 t + \alpha_2)} : \alpha_i \text{ are defined mod } 2\pi \} .$$

By the uniqueness theorem for the Fourier series of an almost periodic function, distinct α_i's mod 2π give rise to distinct elements in $H(q)$. Thus the map

$$h : T^2 \to H(q) : (\alpha_1, \alpha_2) \to a_1 e^{i(\omega_1 t + \alpha_1)} + a_2 e^{i(\omega_2 t + a_2)}$$

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[1] See Fink (1974; p. 58).

is continuous, 1-1 and onto and thus a homeomorphism of topological spaces.
Also it carries the orbits of the dynamical system

$$T^2 \times R \to T^2 \colon ((\alpha_1, \alpha_2), t) \to (\alpha_1 + \omega_1 t, \alpha_2 + \omega_2 t)$$

onto the orbits of the translational flow and is an isomorphism of the topological
groups. The function q corresponds to $(0, 0) \in T^2$ and the orbit of $q, \{q_\tau \colon \tau \in R\}$, corresponds to the dense line $\{(\omega_1 t, \omega_2 t) \colon t \in R\}$ on T^2. See Figure 2.
In general the hull of a quasi periodic function is an s-torus where s is the
dimension of the module S over the integers I, see Pontryagin (1966).
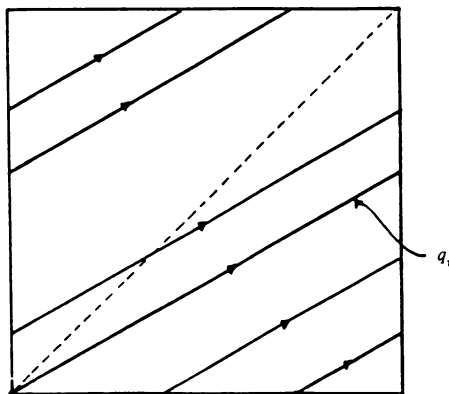

FIGURE 2. Quasi periodic hull: A torus

An interesting extreme case related to the above occurs when the frequencies
$\{\omega_k\}$ are independent over the rationals and not finite. Bohr (1959) calls this
the *disharmonic case*, and he proves that the Fourier series converges uniformly
to the function. The hull in this case is homeomorphic to a countably infinite
dimensional torus, i.e. a countable product of circles with the product topology.

Let us turn next to the limit periodic example (LP). We shall give several
characterizations of the hull $H(\ell)$. Let τ_n be a sequence such that $\ell_{\tau_n} \to \ell^*$
uniformly where

$$(2.3) \qquad \ell^*(t) = \sum_0^\infty a_k e^{i2\pi(t+\phi_k)/2^k} .$$

We may assume that $\tau_n \to \phi_k \bmod 2^k$ as $n \to \infty$ and so $\tau_n \to \phi_{k+l} \bmod 2^{k+l}$,
or

$$(2.4) \qquad \phi_k = \phi_{k+l} \quad \bmod 2^k .$$

We claim that the hull of ℓ is precisely the set of functions ℓ^* as given in
(2.3) where the ϕ_k are defined mod 2^k and satisfy (2.4). Select representatives
so that $0 < \phi_k < 2^k$, then condition (2.4) becomes $\phi_{k+l} = \phi_k + \sigma_k 2^k$ where

$\sigma_k = 0$ or 1. Hence

$$\phi_N = \phi_0 + \sigma_0 + \sigma_1 2^1 + \cdots + \sigma_{N-1} 2^{N-1}$$

Let $\varepsilon > 0$ be given and chose N so large that $\sum_{N+1}^{\infty} |a_k| < \varepsilon$. Then the translate ℓ_{ϕ_N} takes on the form

$$\ell_{\phi_N}(t) = \sum_{k=0}^{N} a_k e^{i2\pi(t+\phi_N)/2^k} + \xi_N(t)$$

$$= \sum_{k=0}^{N} a_k e^{i2\pi(t+\phi_k)/2^k} + \xi_N(t)$$

where $|\xi_N(t)| \leq \varepsilon$. Thus $\ell_{\phi_N}(t) \to \ell^*(t)$ uniformly, consequently the hull of ℓ is the set of all ℓ^* of the form (2.3) with (2.4) holding.

By the uniqueness theorem for the Fourier series of an almost periodic function, distinct ϕ_k satisfying (2.4) give rise to distinct functions. Let $\tilde{\ell}$ denote another function in the hull of ℓ with phase angles $\{\tilde{\phi}_k\}$. Let $\varepsilon > 0$ and N be given and define

$$U(\ell^*, N, \varepsilon) = \{\tilde{\ell} \in H(\ell): |\phi_k - \tilde{\phi}_k| < \varepsilon \text{ for } k = l, \dots, N\}.$$

This set forms a base for the neighborhood system of ℓ^* in $H(\ell)$ since it is clear that if N is large and ε is small then ℓ^* and $\tilde{\ell}$ will be uniformly close. This suggests a coordinate system for $H(\ell)$ as an infinite product $\times_{k=0}^{\infty} S^1$ where $S^1 = \{e^{i\theta}: 0 \leq \theta < 2\pi\}$ is the unit circle in the complex plane and the product has the usual product topology. We set $\theta_k = 2\pi\phi_k/2^k$ so by (2.4) $\theta_k = 2\theta_{k+l}$, mod 2π and we set $z_k = e^{i\theta_k}$ so $z_k = z_{k+l}^2$. Then ℓ^* is given the coordinate $\{z_0, z_1, \dots\} \in S$. Since $z_k = z_{k+l}^2$ the coordinates of ℓ^* are in the inverse limit system

$$(2.5) \qquad\qquad S_2: S^1 \xleftarrow{z^2} S^1 \xleftarrow{z^2} S^1 \leftarrow \cdots .$$

This is the classical coordinate system for the 2-solenoid as given in Hocking and Young (1961).

Let T_0 be a solid torus in a standard embedding in R^3 as given by rotating a meridional disk $D(0) = \{(x, 0, z): (x - 10)^2 + z^2 \leq 1\}$ about the z axis as illustrated in Figure 3. (In Figure 3, $D(0)$ is shaded.) Let ϕ_0 be the polar angle in the (x, y) plane normalized so that ϕ_0 is defined mod 1 and let $D(\phi_0)$ be the image of $D(0)$ after being rotated by ϕ_0. Let T_1 be a solid torus lying within the interior of T_0, longitudinally encircling T_0 twice and with meridional radius $1/4$ as illustrated in Figure 3. Let ϕ_1 be an angular variable on T_1 which measures longitudinal displacement in T_1 and is defined mod 1. As ϕ_1 traverses $[0, 2]$ the meridional disk $D(\phi_1)$ in T_1 encircles the z-axis twice. Note that T_1 intersects $D(\phi_0)$ in two disks $D(\phi_1')$ and $D(\phi_1'')$ where $\phi_0 \equiv \phi_1' \equiv \phi_1'' \mod 1$.
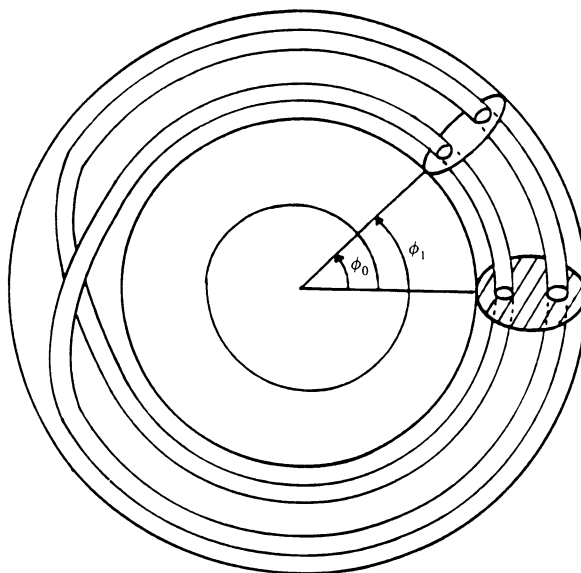
FIGURE 3. Limit periodic hull: A solenoid

Continue in this fashion to define T_{k+1} encircling the torus T_k twice with meridional diameter $1/4^{k+1}$ with longitudinal angular coordinate ϕ_{k+1} defined mod 2^{k+1}. The 2-solenoid $\sum_2$ is simply the intersection $\sum_2 = \bigcap_{k=0}^{\infty} T_k$ which is a nonempty, connected, compact, one-dimensional subspace of R^3 and so is a "Klosed Kurve" in the sense of Menger. However, $\sum_2$ is not locally connected and hence cannot be a topological manifold. It is clear from the above construction that $H(\ell)$, the inverse limit system (2.5), and $\sum_2$ are all homeomorphic.

There is a standard minimal flow on $\sum_2$ which corresponds to the translation flow restricted to $H(\ell)$. The flow is defined by

$$P_t(\ldots, z_k, \ldots) = (\ldots, z_k \exp(i2\pi t/2^k), \ldots)$$

which corresponds to uniform rotation about the z-axis in the solenoid $\sum_2$ as in Figure 3. The solenoid obtains a continuous Abelian group structure by component multiplication in the inverse limit representation, and this corresponds to the general Abelian group structure on the hull of any almost periodic function. The reader is referred to Nemytskii and Stepanov (1960; p. 392) for a careful presentation of the minimal solenoid flow. These examples illustrate the general fact that compact Abelian groups are inverse limits of tori, see Pontryagin (1966).

The solenoid $\sum_2$ constructed above is the simplest of an uncountable number of solenoids which occur naturally in dynamical systems. This simple solenoid can be viewed as the Feigenbaum limit of a period doubling sequence. In the general construction of a solenoid the T_{k+1} solid torus encircles the torus T_k

an arbitrary number of times, giving rise to uncountably many topologically distinct sets. Each of these solenoids carry a minimal flow, which Markus and Meyer (1980) show occur generically in Hamiltonian systems.

II.2 Cross sections and Poincaré mappings. Let $h: D \to D$ be any homeomorphism of a compact metric space D. Then h defines a discrete dynamical system and let $T: D \to R$ be positive and continuous. The parallel flow $\gamma: (D \times R) \times R \to D \times R: ((d, \tau), t) \to (d, \tau + t)$ can be dropped to the quotient space $(D \times R)/\sim$ when $\sim$ is the equivalence relation $(d, t + T(d)) \sim (h(d), t)$. The flow on this quotient space is called the *suspension* of h, see Figure 4.
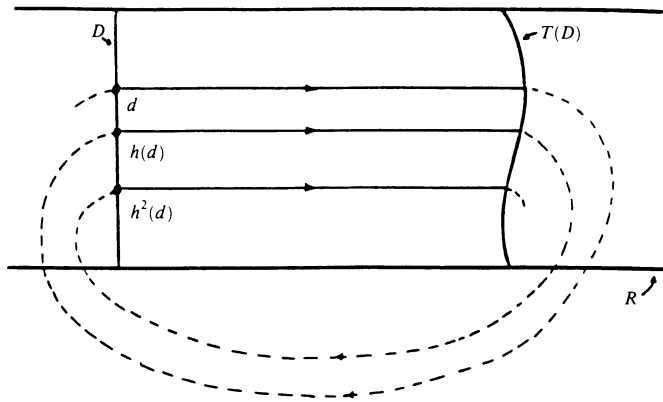

FIGURE 4. Suspension of a diffeomorphism h

A flow $\pi: X \times R \to X$, where X is a compact metric space, admits a *(global)* *cross section* Z if (i) Z is a closed subset of X (ii) all trajectories meet Z and (iii) there is a positive continuous function $T: Z \to R$ such that $\pi(z, T(z)) \in Z$ and $\pi(z, t) \notin Z$ for all $z \in Z$ and $0 < \tau < T(z)$. The function T is called the *first return time*. The *Poincaré map* (or section map) is the map $\eta: Z \to Z: z \to \pi(z, T(z))$, which is a homeomorphism of Z and defines a discrete dynamical system associated with the flow π. Flows that admit global cross sections are precisely suspensions of discrete dynamical systems. See Ikegami (1969) and Neumann (1976) for more properties of cross sections.

A global cross section for a flow need not be connected. For example, if X is a nontrivial periodic orbit, then any nonempty finite subset of X is a global cross section.

The first return time $T: Z \to R$ is a continuous function defined on Z. By changing the time-scale, one can construct an equivalent flow $\hat{\Pi}$ where $T(z) \equiv 1$. While this fact is known, we present the basic ideas here. For $z \in Z$ and an integer k define $\hat{\pi}(z, k) = \eta^k(z)$ where η is the Poincaré map. Next for $\tau \in R$ let k be an integer with $\tau = k + \sigma$ where $\sigma \in [0, 1)$ and define

$$\hat{\pi}(z, \tau) = \pi(\eta^k(z), T(\eta^k(z))\sigma), \qquad z \in Z.$$

One can then show that $\hat{\pi}(z,\tau)$ is well defined and that for any integer ℓ and any $\tau \in R$ one has

$$\hat{\pi}(z,\ell+\tau) = \hat{\pi}(\hat{\pi}(z,\ell),\tau), \qquad z \in Z.$$

Since Z is a global cross section, it follows that for each $x \in X$, there is a unique $z_x \in Z$ and $t_x \in [0,1)$ with $x = \hat{\pi}(z,t_x)$. For $x \in X$ and $\sigma \in R$ one defines

$$\hat{\pi}(x,\sigma) = \hat{\pi}(\hat{\pi}(z_x,t_x),\sigma) = \hat{\pi}(z_x,t_x+\sigma).$$

It is not difficult now to show that $\hat{\pi}$ is a flow on X and that Z is a cross section with first return time $\widehat{T} \equiv 1$ and with Poincaré map η.

The translational flow on the hull of a nonconstant almost periodic function always admits a cross section. Let f be almost periodic and have a Fourier series as in (2.1). Then $g \in H(f)$ has a Fourier series (2.2) and

$$g_\tau(t) \sim \sum a_k e^{i\omega_k(t+\phi_k+\tau)} \sim \sum a_k e^{i\omega_k(\phi_k+\tau)} e^{i\omega_k t}.$$

Thus the Fourier coefficient corresponding to the frequency ω_k is

$$s_k(g_\tau) = M\big(g_\tau(t)e^{-i\omega_k t}\big) = a_k e^{i\omega_k(\phi_k+\tau)},$$

which has a constantly changing argument as τ varies, provided $a_k \neq 0$ and $\omega_k \neq 0$. Since g is nonconstant there is a k with $a_k \neq 0$ and $\omega_k \neq 0$. We restrict now to one such k. Thus a cross section to this translational flow is given by

$$Z = \{g \in H(f): \arg s_k(g) = 0 \text{ or } \phi_k \equiv 0 \bmod 2\pi/\omega_k\}.$$

In this case the first return time $T(z) = 2\pi/\omega_k$ is constant. Thus cross sections and their first return times are intimately connected with the Fourier spectrum of g.

Returning to the quasi periodic example (QP) we note that $q^* \in H(q)$ if and only if $q^*(t) = a_1 e^{i(\omega_1 t+\alpha_1)} + a_2 e^{i(\omega_2 t+\alpha_2)}$. Hence one cross section is $\alpha_1 = 0$ and the first return time is $2\pi/\omega_1$. The angle α_2 is a coordinate on Z, i.e. Z is a circle in the torus. The Poincaré map in this coordinate system is $\eta: \alpha_2 \to \alpha_2 + (\omega_2/\omega_1)2\pi$ which is an irrational rotation of the circle.

For the limit periodic example (LP), let $\ell^* \in H(\ell)$ be given by (2.3). As in the previous example we can define a cross section by requiring that the argument of one of the Fourier coefficients of ℓ^* be zero or equivalently that $\phi_k \equiv 0 \bmod 2_k$ for some fixed k. Consider the cross section Z defined by $\phi_0 \equiv 0 \bmod 1$ (the shaded disk in Figure 3), so the first return time is 1. Clearly the intersection of this disk and $\sum_2$ is a Cantor set. However the associated Poincaré map is more easily understood in the coordinates given by the inverse limit system S_2 in (2.5). In that case $(z_0, z_1, \dots) \in Z \subset S_2$ if and only if $z_0 = 1$. Thus

$$z_1 \in \{1, -1\}, \dots, z_k \in e^{ij2\pi/2^k} : j = 0, 1, \dots, 2^k - 1.$$

The Poincaré map $\eta: Z \to Z$ is

$$(2.6) \qquad\qquad \eta: (\ldots, e^{ij2\pi/2^k}, \ldots) \to (\ldots, e^{i(j+1)2\pi/2^k}, \ldots).$$

In the kth position the free index j ranges over $0 \le j < 2^k$, which suggest using binary notation for j with k binary digits. That is in the kth position one has $j = b_k b_{k-l} \cdots b_1$. where $b_j = 0$ or 1. From (2.6) we see that the Poincaré map η is just addition mod 2^k in the kth position. Thus passing to the limit gives $Z = \times_{k=1}^{\infty} I_2$ with the usual product topology, where I_2 is a set consisting of two points and which is convenient to label as $I_2 = \{1, 2\}$. Then $z \in Z$ can be used as an infinite binary number $z = \cdots b_3 b_2 b_1$, where $(b_j + 1) \in I_2$. The Poincaré map is just binary addition in this coordinate system and so is equivalent to the classical "adding machine" of dynamical systems. The adding machine is one of the simplest examples of a discrete almost periodic dynamical system.

II.3 Bernoulli bundles. For $n \ge 2$, let $I_n = \{1, 2, \ldots, n\}$. Define $B_n = \times_{k=-\infty}^{\infty} I_n$, i.e., B_n is collection of all infinite bisequences on the symbols $s \in I_n$. Thus if $q \in B_n$ then $q = (\ldots, q_{-1}, q_0, q_1 \ldots)$ or more simply written $q = \ldots q_{-1} q_0 . q_1 \ldots$, where the zeroth position is to the left of the decimal point. Let $A: B_n \to B_n$ be the *shift map* or *shift automorphism* defined by $A(q)_i = q_{i+1}$, i.e., A shifts the *decimal point* one place to the right. This is a classical dynamical system (see Morse (1921)) which has found many applications in contemporary dynamical systems, see for example Smale (1963) and Devaney (1986). The discrete dynamical system (B_n, A) is sometimes referred to as the *Bernoulli shift*, or the full shift on n symbols. There are other invariant subshifts which arise naturally in our theory, these are the subsets of finite type, which we define next.

A *transition matrix* is an $n \times n$ matrix $K = \{k_{ij}\}$ with entries which are either 0 or 1. For any transition matrix K we define a subset $B_n(K)$ of B_n by

$$B_n(K) = \{q \in B_n : k_{q_i q_{i+1}} = 1 \text{ for all } i\}.$$

In other words adjacent pairs of entries in a sequence $q \in B_n$ determine a location in the matrix K, the $q_i q_{i+1}$ position. Also the sequence q is in $B_n(K)$ if and only if $k_{q_i q_{i+1}} = 1$ for every i. Notice that $B_n(K)$ is A-invariant, i.e. $AB_n(K) = B_n(K)$. The set $B_n(K)$ is referred to as a *subshift of finite type*. Clearly $B_n(K)$ is a closed invariant subset of B_n.

The transition matrix K serves as a litany of which values q_j may follow a given value q_i for $q \in B_n(K)$ in the sense that one can have q_{i+1} following q_i if and only if $k_{q_i q_{i+1}} = 1$, which we write as $q_i \to q_{i+1}$ for short. If one thinks of the digits $\{1, 2, \ldots, n\}$ as representing states in a finite state automata and K as being its transition matrix, then $q_i \to q_{i+1}$ illustrates the possible state jumps in $B_n(K)$. Alternatively the zeros in the transition matrix K rules out certain adjacent pairs in a sequence $q \in B_n(K)$. For example if K is a transition matrix where every entry is 1, then $(B_n(K), A)$ is the full shift on n symbols.

We will be primarily interested in a specific $n \times n$ matrix given by

$$(2.7) \qquad K = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ & & \vdots & & & \ddots & & \vdots & \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}.$$

That is K has 1's on the first super-diagonal and at the $(n,1)$ and (n,n) positions. Thus one has $1 \to 2$, $2 \to 3$, ..., $n-1 \to n$, $n \to n$, and $n \to 1$, see Figure 5.
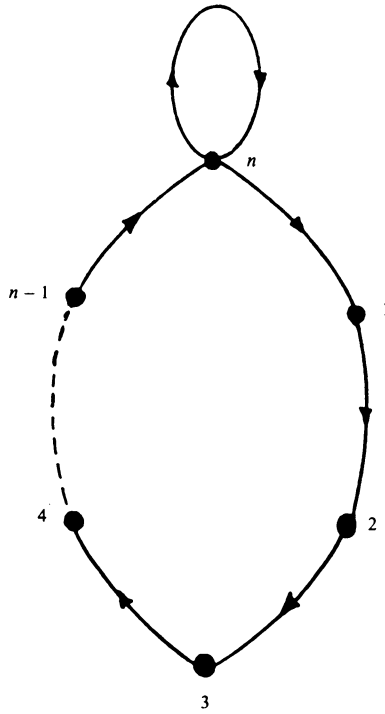

FIGURE 5. State transitions for K given by (2.7)

Let an overbar on a symbol, or on a sequence of symbols, mean that it is to be repeated infinitely often, so $\overline{1}.123\overline{1} = \dots 11.12311 \dots$. The restriction of A to $B_n(K)$, which we denote by A, has a unique fixed point $v^* = \overline{n}.\overline{n}$. Furthermore the point $u^* = \overline{n}.12345 \dots n\overline{n}$ in $B_n(K)$ is doubly asymptotic, or homoclinic to v^*, i.e., $A^t u^* \to v^*$ as $|t| \to \infty$. A point of period n is given by $q = \overline{12 \dots n . 12 \dots n}$. Periodic points of period $p > n$ can be obtained by replacing the single n in $\overline{12 \dots n . 12 \dots n}$ by a finite sequence of $(p+1-n)n$'s. Thus (B_n, A) contains periodic points of every period $\geq n$.

Since $B_n(K)$ is invariant under A, it is also invariant under various powers of A. One can readily compute the transition matrices for A^2, A^3, etc. For example if K is given by (2.7) with $n = 4$, then every entry in the transition matrix for A^6 is 1. In other words, $(B_4(K), A^6)$ is a full shift on 4 symbols. More generally one can show that $(B_n(K), A^{2n-2})$ is a full shift on n symbols when K is given by (2.7).

The Bernoulli shift flow on B_n is the prototypical example of chaotic behavior. Even though this is a completely deterministic dynamical event, it contains important features of randomness. The simplest way to see this is to take the view point of an observer who is able to watch a single coordinate of $q \in B_n$, say q_0, the entry to the left of the decimal point. After applying A to q the observer sees q_1. Since the observer does not know which $q \in B_n$ the shift mapping A is operating on, the entry q_1 which appears after applying A seems to be completely random. This form of randomness can be made more precise and we refer the reader to Sinai (1973), Alekseev (1976), Holmes (1980), Cornfeld, Fomin and Sinai (1982), and Marsden (1984) for further information.

Our next objective is to define a Bernoulli bundle $\mathscr{B}_n$ for $n \geq 2$. This will be a fiber bundle over a space X where the fibers are B_n. More precisely let π be a flow on X with section Z, and let $\eta : Z \to Z$ denote the Poincaré map with first return time $T : Z \to R$. Let K be the $(n \times n)$ transition matrix (2.7). The product of A and η is simply

(2.8) $\qquad A \times \eta : B_n(K) \times Z \to B_n(K) \times Z : (q, z) \to (A(q), \eta(z))$,

The *Bernoulli bundle* $\mathscr{B}_n(K)$ is formed by projecting the parallel flow

(2.9) $\qquad \Psi : (B_n(K) \times Z \times R) \times R \to B_n(K) \times Z \times R$
$$: ((q, z, \tau), t) \to (q, z, \tau + t)$$

onto the quotient space

$$\mathscr{B}_n(K) = B_n(K) \times Z \times R / \sim$$

where $(q, z, t + T(z)) \sim (A(q), \eta(z), t)$.

The flow on the Bernoulli bundle $\mathscr{B}_n(K)$ is given by (2.9) and we shall write

(2.10) $\qquad\qquad\qquad \mathscr{B}_n(K) = B_n(K) \otimes X$.

and denote the flow itself by $A \otimes \eta$. The notation in (2.10) emphasizes that $(\mathscr{B}_n(K), A \otimes \eta)$ is a skew product flow over (X, π) and that the mapping

$$P_2 : (q, z, \tau) \to \pi(z, \tau)$$

is a flow homomorphism,[2] see Figure 6. Let $S \subset B_n(K)$ and $M \subset X$, and define

$$S \otimes M = \{(s, m, \tau) \in \mathscr{B}_n(K) : s \in S, m \in M, \tau \in R\}.$$

Then $S \otimes M$ denotes the orbit in $\mathscr{B}_n(K)$ generated by (S, M).

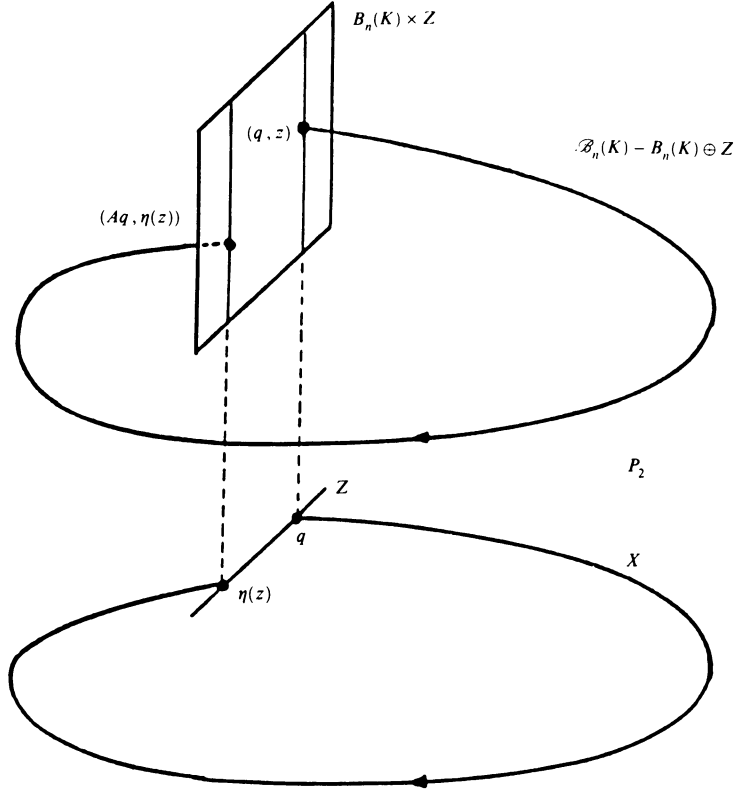[2] Most of the conclusions in the remainder of this section follow from Sacker and Sell (1977).

FIGURE 6. The Bernoulli bundle flow

If $q_0 \in B_n(K)$ is a fixed point of A, i.e., if $q_0 = \bar{n}.\bar{n}$, then $\{q_0\} \times X$ is a 1-cover of X. More generally if $q_1 \in B_n(K)$ is a periodic point of A of minimal period p, then $\{q_1\} \otimes X$ is a p-cover of X. If, in addition, X is an almost periodic minimal set in the flow π, then $\{q_1\} \otimes X$ is an almost periodic minimal set in the flow Ψ on $\mathscr{B}_n(K)$.

The Bernoulli shift space B_n is metrizable. More specifically if $p, q \in B_n$ then

$$d(p, q) = \sum_{m=-\infty}^{\infty} 2^{-|m|}|p_m - q_m|$$

is a metric on B_n. The shift flow A on B_n is uniformly distal (or expansive). Indeed if $p, q \in B_n$ with $p \neq q$ then there is an integer k such that $d(A^k p, A^k q) \geq 1$. As a result the flow $A \otimes \eta$ on the Bernoulli bundle $\mathscr{B}_n(K)$ is distal whenever the flow π on X is distal.

One interesting application of the distality property occurs when X is an almost periodic minimal set. In this case, if M is a closed invariant set in $\mathscr{B}_n(K)$ with the property that for some $x \in X$ the fiber $P_2^{-1}(x)$ in $\mathscr{B}_n(K)$ is

finite, then M is the union of a finite number of almost periodic minimal sets, each of which is a finite covering space of X.

III. The Melnikov transform for almost periodic systems

III.1 The induced skew product flow. Consider the differential equation

$$(3.1) \qquad \dot{x} = F(x) + \varepsilon f(t, x)$$

where $x \in R^2$, and $F \colon R^2 \to R^2$ is smooth and $f \in \mathscr{G}$. Here $\mathscr{G}$ denotes the collection of all functions $f \colon R \times R^2 \to R^2$ with the property that for every compact set $K \subset R^2$, (i) the function f is uniformly continuous on $R \times K$ and (ii) there is a constant k such that

$$(3.2) \qquad |f(t, x) - f(t, y)| \le k|x - y|, \qquad t \in R, \; x, y \in K.$$

For many of our applications we will assume that $f \in \mathscr{A}\mathscr{P}$, where $\mathscr{A}\mathscr{P}$ consists of those functions from $\mathscr{G}$ which are almost periodic in t uniformly for x in compact sets in R^2.

On $\mathscr{G}$ we shall use the compact open topology, which means that a sequence $\{f_n\}$ converges to f if and only if $f_n \to f$ uniformly compact subsets of $R \times R^2$. As before the translation of a function $f \in \mathscr{G}$ is given $f_\tau(t, x) = f(\tau + t, x)$ and the hull $H(f)$ is

$$H(f) = Cl\{f_\tau \colon \tau \in R\}.$$

One can show that if $f \in \mathscr{G}$ then $H(f)$ is a compact invariant set and $H(f) \subset \mathscr{G}$, cf. Sell (1967).

The unperturbed system when $\varepsilon = 0$ is assumed to be Hamiltonian, and consequently, the trace of the Jacobian of F is identically zero. (As a result the flow for the unperturbed equation is area-preserving in R^2.) Furthermore, the unperturbed system is assumed to have a nondegenerate saddle point at $v_0 \in R^2$ and an orbit $u^0(t)$ homoclinic to v_0, i.e. $u^0(t) \to v_0$ as $t \to \pm\infty$. Thus the stable and unstable manifolds of v_0 intersect along the orbit of u^0. See Figure 1.

Let K be a large closed disk in R^2 which contains v_0 and the orbit $u^0(t)$ in its interior. For $\ell_0 > 0$ we define

$$\mathscr{L} = \{f \in \mathscr{G} \colon |f(t, x)| \le \ell_0, \text{ for } t \in R, x \in K\}.$$

Let $0 < \varepsilon_0$ be given. For $|\varepsilon| \le \varepsilon_0$ we let $\phi(t, x_0, f, \varepsilon)$ be the solution of (3.1) that satisfies $\phi(0, x_0, f, \varepsilon) = x_0$. We consider F as fixed and so suppress the dependence of the solution ϕ on F for simplicity. However f will be taken from a translation invariant compact subset $H \subset \mathscr{L}$, which oftentimes will be the hull of some almost periodic function.

In order to better understand the underlying geometry of (3.1) it is useful to introduce the concept of a skew product flow. Let H be a translation invariant

subset of $\mathscr{L}$. Assume, for simplicity, that the solution [3] $\phi(t, x_0, f, \varepsilon)$ of (3.1) is defined for all t when $|\varepsilon| < \varepsilon_0$ and $f \in H$. The skew product flow on $R^2 \times H$ is given by

$$\Pi: R \times (R^2 \times H) \to (R^2 \times H): (t, x_0, f) \to (\phi(t, x_0, f, \varepsilon), f_t),$$

see Miller (1965) and Sell (1967)

The next theorem, which is a classical result, describes how the hyperbolic fixed point v_0 behaves when ε becomes nonzero.

Theorem 0. *There is an ε_0, $0 < \varepsilon_0 \leq 1$, such that for all $f \in \mathscr{L}$, $|\varepsilon| < \varepsilon_0$ there is a unique bounded solution $v(t, f, \varepsilon)$ of (3.1) that satisfies*

$$\sup_{f \in \mathscr{L}} \sup_{t \in R} |v(t, f, \varepsilon) - v_0| \to 0, \quad as \ \varepsilon \to 0.$$

Furthermore, v satisfies

(3.3) $\qquad v(t, f_\tau, \varepsilon) = v(t + \tau, f, \varepsilon) = v_\tau(t, f, \varepsilon), \qquad t, \tau \in R.$

In particular, one has $v(\tau, f, \varepsilon) = v(0, f_\tau, \varepsilon)$. Also the mapping $f \to v(\cdot, f, \varepsilon)$ is a continuous mapping of $\mathscr{L}$ into $\mathscr{C}(R, R^n)$, where the latter space has the compact open topology. If $f \in \mathscr{A}\mathscr{P}$, then $v(t, f, \varepsilon)$ is also almost periodic in t, and the frequency module of v is contained in the frequency module of f.

Proof. The proof of the existence of $v(t, f, \varepsilon)$ and the continuity in f and ε follows from the Lyapunov-Perron method using integral equations and involving exponential dichotomies, see Coppel (1965) or Hale (1969) for example. The proof of (3.3) is an immediate consequence of the uniqueness of v. Indeed both $v(t, f_\tau, \varepsilon)$ and $v(t + \tau, f, \varepsilon)$ are bounded solutions of the translated equation

$$x' = F(x) + \varepsilon f_\tau(t, x).$$

The conclusion on the almost periodic behavior follows from Sacker and Sell (1977). □

With $\overline{x}(f, \varepsilon) = v(0, f, \varepsilon)$ one has $\phi(t, \overline{x}(f, \varepsilon), f, \varepsilon) = v(t, f, \varepsilon)$. Next define $V = \{(x(f, \varepsilon), f) \in R^2 \times H: f \in H\}$, where H is a translation invariant subset of $\mathscr{L}$. The identity (3.3) shows that V is an invariant set for the flow Π since

$$\phi(t, \overline{x}(f, \varepsilon), f, \varepsilon) = v(t, f, \varepsilon) = v(0, f_t, \varepsilon) = \overline{x}(f_t, \varepsilon).$$

The mapping $Q: H \to V: f \to (\overline{x}(f, \varepsilon), f)$ is a flow homomorphism by (3.3) also. In other words, the skew product flow defined by (3.1) has for small ε an invariant set V which is flow homomorphic to the translation flow on H.

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[3] Since we shall only be interested in the behavior of solutions of (3.1) in the compact set K, there is no loss of generality in assuming the solutions of (3.1) to be defined for all $t \in R$. If this were not the case one could modify F and f near $x = \infty$, to accomplish this global existence property.

III.2 The stable and unstable bundles. Hale (1969) and Sell (1978) establish the local stable and unstable manifold theorem for (3.1). We will consider the global stable and unstable manifolds as well, since we are assuming the global existence of solutions of (3.1). The global version is obtained from the local version by the standard argument of backward and forward integration under the assumption that the solutions of (3.1) are defined for all t. (See for example Palis and de Melo (1980) for a discussion of using backward integration to obtain the global stable manifold from the local stable manifold for autonomous equations.) Since this argument has a straightforward extension to the skew product flows we consider here, we will apply it without further comment to our case.

Recall that the global stable manifold $W^s(f, \varepsilon)$ is defined as

$$(3.4) \qquad W^s(f, \varepsilon) = \{x_0 \colon \|\phi(t, x_0, f, \varepsilon) - v(t, f, \varepsilon)\| \to 0 \text{ as } t \to +\infty\}.$$

The local stable manifold theorem states that there is a sufficiently small $\delta > 0$ such that

$$(3.5) \quad W^s_{\text{loc}}(f, \varepsilon) \overset{\text{def}}{=} \{x_0 \in W^s(f, \varepsilon) \colon \|\phi(t, x_0, f, \varepsilon) - v(t, f, \varepsilon)\| < \delta \text{ for } t \geq 0\}$$

is a smooth one-dimensional manifold. In particular, there is a smooth function

$$w^s_{\text{loc}} \colon (-\delta, \delta) \times H \times (-\varepsilon_0, \varepsilon_0) \to R^2$$

such that

$$W^s_{\text{loc}}(f, \varepsilon) = \{w^s_{\text{loc}}(\sigma, f, \varepsilon) \colon \sigma \in (-\delta, \delta)\} = \text{ Graph } w^s_{\text{loc}}(\cdot, f, \varepsilon),$$

and for fixed $(f, \varepsilon) \in H \times (-\varepsilon_0, \varepsilon_0)$ the map

$$w^s_{\text{loc}}(\cdot, f, \varepsilon) \colon (-\delta, \delta) \to R^2$$

is an embedding of the interval $(-\delta, \delta)$ into R^2. The function w^s_{loc} is smooth in all its arguments. By backward integration w^s_{loc} can be extended to

$$w^s \colon R \times H \times (-\varepsilon_0, \varepsilon_0) \to R^2$$

where the restriction of w to $(-\delta, \delta) \times H \times (-\varepsilon_0, \varepsilon_0)$ is w^s_{loc} and so that the global stable manifold $W^s(f, \varepsilon)$ is the graph of w^s, i.e.,

$$W^s(f, \varepsilon) = \{w^s(\sigma, f, \varepsilon) \colon \sigma \in R\}.$$

For fixed (f, ε) the map

$$w^s(\cdot, f, \varepsilon) \colon R \to R^2$$

is an immersion of the line. For the skew product flow we define the *stable bundle* by

$$\mathscr{W}^s(\varepsilon) = \{(W^s(f, \varepsilon), f) \colon f \in H\}.$$

As was done in (3.3) one can establish that

$$\phi(\tau, W^s(f, \varepsilon), f, \varepsilon) = W^s(f_\tau, \varepsilon),$$

i.e., $\mathscr{W}^s(\varepsilon)$ is an invariant set for the skew product flow Π. Furthermore, $\mathscr{W}^s(\varepsilon)$ which is characterized by the formula

$$(3.6) \qquad \mathscr{W}^s(\varepsilon) = \{p \in R^2 \times H \colon \Pi(t,p) \to V \text{ as } t \to +\infty\}.$$

W_{loc}, W^s and $\mathscr{W}^s$ are called respectively the local stable manifold, the stable manifold and the stable bundle. By replacing $t \geq 0$ by $t \leq 0$ in (3.5) and $t \to +\infty$ by $t \to -\infty$ in (3.4) and (3.6) one defines the corresponding unstable sets W_{loc}^u, W^u, $\mathscr{W}^u$.

Let $p = (x_0, f) \in \mathscr{W}^s(\varepsilon)$. We define the (partial) tangent space by

$$(3.7) \qquad T_p \mathscr{W}^s(\varepsilon) = \operatorname{span}\left\{ \frac{\partial w^s}{\partial \sigma}(\sigma_0, f, \varepsilon) \right\}$$

where $x_0 = w^s(\sigma_0, f, \varepsilon)$. That is we consider only the component of the tangent space which lies in the plane R^2.

If $p \in \mathscr{W}^s(\varepsilon) \cap \mathscr{W}^u(\varepsilon)$ then we say p is a *homoclinic point* (homoclinic to V) and $\{\Pi(t,p) \colon t \in R\}$ is a *homoclinic orbit*. Thus $\Pi(t,p) \to V$ as $t \to \pm\infty$. If $p \in \mathscr{W}^s(\varepsilon) \cap \mathscr{W}^u(\varepsilon)$ and $T_p R^2 = R^2 = T_p \mathscr{W}^s(\varepsilon) + T_p \mathscr{W}^u(\varepsilon)$, then we say $\mathscr{W}^s(\varepsilon)$ and $\mathscr{W}^u(\varepsilon)$ *intersect transversally* at p. If at each point $p \in \mathscr{W}^s(\varepsilon) \cap \mathscr{W}^u(\varepsilon)$, $\mathscr{W}^s(\varepsilon)$ and $\mathscr{W}^u(\varepsilon)$ intersect transversally at p, then we say $\mathscr{W}^s(\varepsilon)$ and $\mathscr{W}^u(\varepsilon)$ intersect transversally and write $\mathscr{W}^s(\varepsilon) \pitchfork \mathscr{W}^u(\varepsilon)$.

III.3 The Melnikov transform.

The Melnikov functional gives a criterion for the existence of transversal homoclinic orbits. At a point $\bar{u} = u^0(0)$ on the homoclinic orbit of the unperturbed system we construct a normal line n perpendicular to $F(\bar{u})$ and measure the separation of the stable and unstable manifolds on this normal, see Figures 1 and 7. We differ from the presentation in Melnikov (1963), Guckenheimer and Holmes (1983) and others by fixing the normal n and measuring the separation at the epoch $t = 0$. While this may appear to be restrictive, we will see that this is the correct approach for generalizing the Melnikov method to nonperiodic perturbations. It is important to note that we allow f to vary over the compact invariant set H.

For $f \in H$ and ε small there are unique solutions

$$u_s(t, f, \varepsilon) = u^0(t) + \varepsilon u_s^1(t, f) + O(\varepsilon^2),$$
$$u_u(t, f, \varepsilon) = u^0(t) + \varepsilon u_u^1(t, f) + O(\varepsilon^2)$$

such that

$$u_s(t, f, \varepsilon) \to v(t, f, \varepsilon) \quad \text{as } t \to +\infty,$$
$$u_u(t, f, \varepsilon) \to v(t, f, \varepsilon) \quad \text{as } t \to -\infty,$$

$u_s(0, f, \varepsilon) \in n$ and $u_u(0, f, \varepsilon) \in n$. Thus u_s and u_u are the unique solutions on the stable and unstable manifold which at $t = 0$ lie on the normal n, cf. Figure 7.
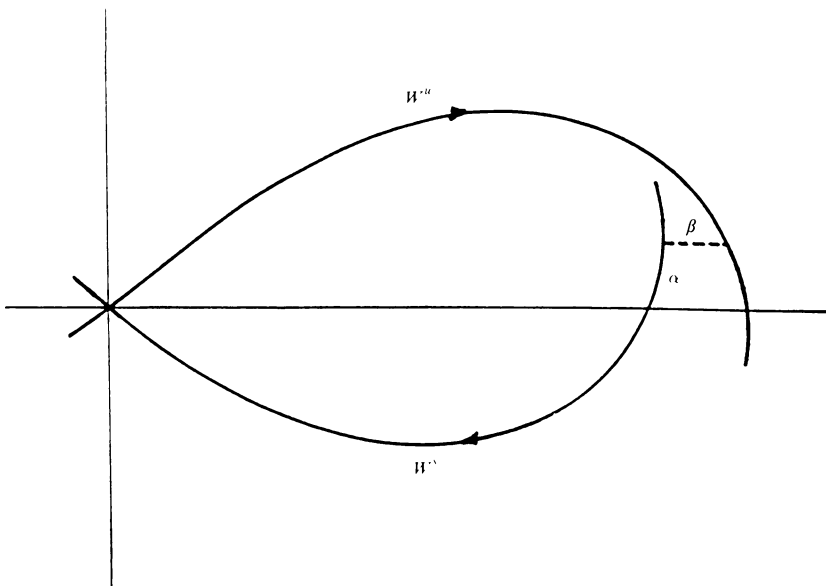
FIGURE 7. Stable manifold coordinates

We define the separation of the stable and unstable manifold on n to be

$$d(f,\varepsilon) = \|u_s(0,f,\varepsilon) - u_u(0,f,\varepsilon)\|$$

where $\|\cdot\|$ denotes the Euclidean norm on R^2. Equivalently one has

$$d(f,\varepsilon) = \mathrm{dist}(W^s(f,\varepsilon) \cap n, W^u(f,\varepsilon) \cap n).$$

By the standard argument,[4] one finds that

$$d(f,\varepsilon) = \varepsilon \nu M(f) + O(\varepsilon^2)$$

where $\nu = \|F(\overline{u})\|^{-1}$ is constant, and M is the Melnikov functional

$$(3.8) \qquad M(f) = \int_{-\infty}^{\infty} F(u^0(t)) \wedge f(t,u^0(t))\,dt.$$

Since $M(f)$ is linear in f, it is differentiable in $f \in \mathscr{G}$. By the standard argument one can show that the function $d(f,\varepsilon)$ is also differentiable in $f \in \mathscr{G}$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ That is,

$$d: H \times (-\varepsilon_0, \varepsilon_0) \to R$$

is a C^1-function. Let $D_1 d$ be the partial of d with respect to its first argument so

$$D_1 d(f,\varepsilon)(g) = \varepsilon \nu M(g) + O(\varepsilon^2).$$

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[4] Melnikov (1963) and Guckenheimer and Holmes (1983) are good references to the classical Melnikov theory used in this section.

In the usual development in the periodic case the Melnikov functional depends on a real parameter. To see how our definition is the natural extension, let $H = \{p_{-\tau} : \tau \in R\}$ be the hull of a periodic function p. The restriction of M to H is

$$M(p_{-\tau}) = \int_{-\infty}^{\infty} F(u^0(t)) \wedge p(t - \tau) \, dt,$$

which is the usual Melnikov function in the periodic case. By allowing f to vary over H, we see that our formulation includes the traditional version.

Since $F(u^0(t))$ tends to zero exponentially fast as $t \to \pm\infty$, we see that $M : \mathcal{G} \to R$ as a bounded linear functional, which we call the Melnikov functional. Given $f \in \mathcal{G}$, we define the composition $\mathcal{M}(f)(t) = M(f_t)$, to be the *Melnikov transform* of f. Note that

$$(3.9) \quad \begin{aligned} \mathcal{M}(f)(t) = M(f_t) &= \int_{-\infty}^{\infty} F(u^0(s)) \wedge f(s + t, u^0(s)) \, ds \\ &= \int_{-\infty}^{\infty} F(u^0(\sigma - t)) \wedge f(\sigma, u^0(\sigma - t)) \, d\sigma. \end{aligned}$$

Since $\mathcal{M}$ is a composition, it follows that if f is almost periodic, then $\mathcal{M}(f)$ is also almost periodic.

In order to prove that the stable and unstable bundles intersect transversally we will require a sign change near some (f, ε). This can be accomplished for small ε by studying the zero set of the Melnikov functional M, which we define to be

$$(3.10) \qquad\qquad Z = \{f \in H : M(f) = 0\}.$$

We will say that the compact invariant set H admits a *simple* Melnikov zero set Z, if Z is nonempty and

$$(3.11) \qquad\qquad \frac{d}{dt} M(f_t)\big|_{t=0} \neq 0$$

for all $f \in Z$. By (3.9) this is equivalent to the Melnikov transform of f having a simple zero at zero. Also by the second part of (3.9) it is clear that the derivative in (3.11) exists even if the original function f is differentiable in x but only continuous in t.

Later in this paper we will derive our main result which asserts that if H admits a simple Melnikov zero set Z, then the skew product flow generated by (3.1) for $f \in H$, contains a Bernoulli bundle $\mathcal{B}_n(K)$ for every n sufficiently large and for every small ε. However we first want to derive an important consequence of the existence of a simple zero set.

Theorem 1. *Let H be a compact invariant set in $\mathcal{G}$. Assume that H is a minimal set and that H admits a simple (nonempty) zero set Z. Then Z is a global cross section to the flow on H.*

Proof. Since H is compact and the Melnikov functional $M : H \to R$ is continuous, it follows that Z is closed and therefore compact. By (3.11) if $f \in Z$

there is a $\delta > 0$ such that $f_t \notin Z$ for $0 < |t| \leq \delta$, and by the compactness of Z, δ can be chosen independent of $f \in Z$. Let $f \in Z$ and $g \in H$. By the above there is a $\delta > 0$ such that $M(f_{-\delta})$ and $M(f_\delta)$ are of opposite sign. Since the orbit of g is dense in H there is a time τ such that g_τ is arbitrary close to f. In particular we can find τ so that $M(g_{\tau-\delta})$ and $M(g_{\tau+\delta})$ are of opposite sign. Therefore for some $\tau_0 \in (\tau - \delta, \tau + \delta)$, one has $g_{\tau_0} \in Z$. This proves that all orbits meet Z.[5] For $f \in Z$ define $T(f) > 0$ to be the least time of return. The above argument also shows that T is a continuous mapping, and thus Z is a global cross section. $\square$

The fact that the zero set Z is a global cross section to the flow on H means that the Melnikov functional prescribes a synchronization—or internal clock—for the flow on H.

III.4 Proper Melnikov perturbations. One of the main objectives in developing dynamical systems is to seek a theories which have some degree of robustness. In other words, one would like to develop a theory which will persist under an appropriate set of perturbations. Not all dynamical theories have this property, however many of the dynamical features we study in this work do. In order to do this it is necessary to present the concept of a proper Melnikov perturbation. Before doing this though let us look at an illustrative example.

Consider the equation

$$(3.12) \qquad x' = F(x) + \varepsilon f(x, \theta), \quad \theta' = \omega,$$

where $x \in R^2$, $\theta \in T^2$ (the 2-dimensional forms), $F: R^2 \to R^2$ and $f: R^2 \times T^2 \to R^2$ are smooth functions, F satisfies the conditions stated in §III.1 and $\omega = (\omega_1, \omega_2)$ is constant with ω_1/ω_2 irrational. This is an example of a quasi periodic perturbation of $x' = F(x)$, and the theory we describe below will apply in this case.

Let us now introduce a small change in (3.12) by perturbing the θ-equation to obtain

$$(3.13) \qquad x' = F(x) + \varepsilon f(x, \theta), \quad \theta' = \omega + \varepsilon g(x, \theta)$$

where $g: R^2 \times T^2 \to T^2$ is smooth. The perturbation theory given in §§III.1 and III.2 still apply in this case, see Hale (1969). In particular the invariant torus $H_0 = \{v_0\} \times T^2$ for the unperturbed equation ($\varepsilon = 0$) becomes an invariant torus H_ε for the perturbed equation when ε is small and nonzero. However the flow on M_ε need not be quasi periodic. In fact it is generally not quasi periodic for $\varepsilon \neq 0$ because one expects a locking-in to occur and that the flow on the torus H_ε will contain stable and unstable periodic orbits. Even though this example is not minimal, it is evident that the flow on H_ε still has a global cross section.

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[5] By using the Birkhoff recurrence property for minimal sets, we see that every orbit meets Z infinitely often.

Because of Theorem 1 and the considerations of the last paragraph we make the following definition: Consider equation (3.1) for $f \in H$, where H is a compact invariant set in $\mathscr{G}$. Let $M: H \to R$ be the Melnikov functional given by (3.8). We shall say that H is a *proper Melnikov perturbation* if there is a global cross section Z to H such that $Z \subset M^{-1}(0)$, i.e., Z lies in the zero-set of the Melnikov functional, and for each $f \in Z$, inequality (3.11) is valid.

The following proposition shows that the property that compact invariant set $H \subset \mathscr{G}$ be a proper Melnikov perturbation is an *open* property in the class of compact invariant sets in $\mathscr{G}$. We shall use the symbol $B_\delta(H)$ to denote an δ-neighborhood of H in $\mathscr{G}$.

Proposition. *Let H be a compact invariant set in $\mathscr{G}$ which is a proper Melnikov perturbation. Then there is an $\delta > 0$ with the property that every compact invariant set $\widehat{H}$ satisfying*

$$(3.14) \qquad \widehat{H} \subset B_\delta(H), \qquad H \subset B_\delta(\widehat{H})$$

is also a proper Melnikov perturbation.

The proof of this is straightforward application of the Implicit Function Theorem and we omit the details.

In the case of equation (3.13) one can show that for ε sufficiently small that (3.14) is valid where $H = H_0$ and $\widehat{H} = H_\varepsilon$. A special case of this is discussed in §VI.

III.5 Duffing's equation. Let us consider one of the standard examples in this subject, Duffing's equation with negative linear stiffness and almost periodic forcing:

$$(3.15) \qquad \dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 - x_1^3 + \varepsilon f(t),$$

where $x = (x_1, x_2)$, $F(x) = (x_2, x_1 - x_1^3)$ and $f: R \to R$ is almost periodic. Here we assume that f is independent of x, i.e., f does not add dissipation to Duffing's equation. In §6 we shall consider a more complicated perturbation of Duffing's equation.

When $\varepsilon = 0$ this system has centers at $(\pm 1, 0)$ and a nondegenerate saddle at the origin. The system is Hamiltonian with $H = \frac{1}{2}(x_2^2 - x_1^2) + x_1^4/4$ and so the phase portrait is easy to analyze, see Figure 1. There are two orbits homoclinic to the origin $v_0 = (0, 0)$. The homoclinic orbits which cross the x_1-axis at epoch $t = 0$ are $\pm(\sqrt{2} \operatorname{sech} t, -\sqrt{2} \operatorname{sech} t \tanh t)$. Choose the right homoclinic orbit by taking the plus sign above. Since the Melnikov functional is linear in f we need only compute it on sines and cosines. Residue calculus yields

$$(3.16) \qquad M(\cos(\omega t + a)) = \sqrt{2}\pi \operatorname{sech}(\pi\omega/2) \sin a.$$

Return to the example (QP) we let H be the hull of the quasi periodic function

$$q(t) = a_1 \cos \omega_1 t + a_2 \cos \omega_2 t$$

where a_1, a_2 are nonzero and ω_1/ω_2 is irrational. Then by the results of the last section, the hull of f is $H = \{a_1 \cos(\omega_1 t + \phi_1) + a_2 \cos(\omega_2 t + \phi_2)\}$, which is homeomorphic to $T^2 = R^2/(2\pi Z)^2$. By (3.16) one has

$$M(a_1 \cos(\omega_1 t + \phi_1) + a_2 \cos(\omega_2 t + \phi_2)) = A_1 \sin\phi_1 + A_2 \sin\phi_2$$

where

$$A_i = a_i \sqrt{2}\, \omega_i \mathrm{sech} \left(\frac{\pi\omega_i}{2} \right).$$

For fixed $\omega_1, \omega_2 \neq 0$, it is clear that we can prescribe A_1 and A_2 and uniquely determine a_1 and a_2. Thus we can first choose the Melnikov function and hence its zero set and then find the function f, i.e., we can invert the Melnikov transform. Thus the zero set Z is defined by $A_1 \sin\phi_1 + A_2 \sin\phi_2 = 0$ and condition (3.11) becomes $\omega_1 A_1 \cos\phi_1 + \omega_2 A_2 \cos\phi_2 \neq 0$.

This example is instructive since one can give explicit conditions for the zero set to be simple. We are interested in the nontrivial case where the parameters are nonzero, so we introduce the dimensionless parameters $A = A_2/A_1$ and $\omega = \omega_1/\omega_2$. In this case, the zero set is always nonempty, so let us assume that there is a point where the condition (3.11) is not satisfied. One then has $\sin\phi_1 = -A \sin\phi_2$ and $\omega \cos\phi_1 = -A \cos\phi_2$. Squaring these equations and eliminating the cosines gives

$$\sin^2\phi_1 = (\omega^2 - A^2)/(\omega^2 - 1), \qquad \sin^2\phi_2 = (\omega^2 - A^2)/A^2(\omega^2 - 1).$$

From these formulas we see that M has a nonsimple zero if and only if $\omega^2 \leq A^2 \leq 1$ or $1 \leq A^2 \leq \omega^2$. On the boundary of this region, i.e., when $A^2 = 1$ or $\omega^2 = A^2$, the zero set of M contains a global cross section to the flow on T^2. In the A, ω parameter space, the set W of parameter values which give rise to a simple zero set is precisely the set where $A^2 > \max(1, \omega^2)$ or $A^2 < \min(1, \omega^2)$. W is a large open set which is not dense.

Next turn to the limit periodic example (LP) and let H be the hull of

$$(3.17) \qquad \ell(t) = \sum_{k=1}^{\infty} a_k \cos 2\pi(t/2^k).$$

Thus H consists of all functions of the form

$$(3.18) \qquad \ell^*(t) = \sum_{k=1}^{\infty} a_k \cos 2\pi((t + \phi_k)/2^k)$$

with $\phi_k = \phi_{k+1} \bmod 2^k$. Now by (3.16) one has

$$(3.19) \qquad M(\ell^*) = \sum_{k=1}^{\infty} A_k \sin 2\pi(\phi_k/2^k)$$

where

$$A_k = \sqrt{2}\, \pi a_k \mathrm{sech}\, (\pi^2 2^k).$$

Here we cannot prescribe the A_k's completely arbitrary as before, but is clear that we can choose nonzero A_k's such that the series (3.17), (3.18) and (3.19) all converge rapidly. By (3.16) one has

$$(3.20) \qquad g(\tau) \stackrel{\text{def}}{=} M(\ell_\tau^*) = \sum_{k=1}^\infty A_k \sin 2\pi((\tau + \phi_k)/2^k),$$

i.e., g is a limit periodic function and the Melnikov transform of ℓ^*. From (3.20) we see that $\ell^* \in Z$ if and only if $g(0) = 0$. Furthermore the first return time for ℓ^* is the first positive zero of g. The condition (3.11) reduces to the assertion that the Melnikov transform g have simple zeros for every $\ell^* \in H$.

By taking $A_1 = 1$ and the remaining A's small, it is clear that one can construct a limit periodic function g of the form (3.20) such that it and everything in its hull has simple zeros. Moreover, the A's can be chosen so small that the corresponding series (3.17) converges uniformly so that ℓ is a limit periodic function.

III.6 Transversal intersections. If $Z \subset H$ is a cross section for the translational flow π on H with first return time $T: Z \to R$, then $S = R^2 \times Z$ is a cross section for the skew product flow Π on $R^2 \times H$ with the same first return time, and with Poincaré map given by

$$(3.21) \qquad \Psi = (\psi, \eta): S \to S: (x^0, f) \to (\psi(x_0, f, \varepsilon), \eta(f)),$$

where

$$\psi(x_0, f, \varepsilon) = \phi(T(f), x_0, f, \varepsilon), \quad \eta(f) = f_{T(f)}.$$

Note that Ψ is a discrete skew product dynamical system on $R^2 \times Z$ for $|\varepsilon| < \varepsilon_0$.

When $\varepsilon = 0$ the solutions of (3.1) do not depend on f and the skew product becomes an ordinary product. For example if f is quasi periodic in t, then the flow Π is the product of the irrational twist flow on the torus, as pictured in Figure 2, and the Duffing flow as pictured in Figure 1. In the limit periodic case the flow Π is the product of the solenoidal flow and the Duffing flow.

An alternate view is given by the Poincaré map (3.21). First consider the Example (QP) with $f = q$ when $\varepsilon = 0$. The cross section of the translational flow is a circle, for example the diagonal of Figure 2, and the Poincaré map on H is an irrational rotation of the circle with return time T. Now integrate Duffing's equation for a time T to obtain a map of the plane as shown in Figure 1. Figure 8 tries to illustrate the product map (3.21) at $\varepsilon = 0$. The α-axis coming out of the plane of the paper is an angular variable and should be identified mod 2π since the space is $R^2 \times S^1$. The map carries a plane perpendicular to the α-axis into another such plane. (This is true for small $\varepsilon \neq 0$ also since the second term in (3.21) is always the translational flow). The α-axis is an invariant circle for the Poincaré map Ψ, and its stable and unstable
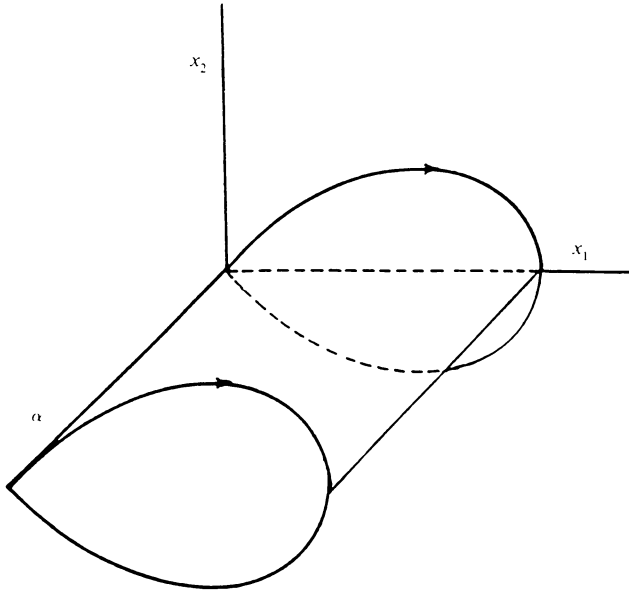
FIGURE 8. The unperturbed dynamics

manifolds are the products of the figure eight and a circle. One of the two loops of the figure eight is illustrated in Figure 8.

For the limit periodic example (LP) with $f = \ell$ and $\varepsilon = 0$, one must use some imagination. Think of a Cantor set along the α-axis in Figure 8. The Poincaré map Ψ is similar to the above except the Cantor set of planes perpendicular to the α-axis are shuffled by the adding machine map.

The main result of this section is the following:

Theorem 2. *Let* (3.1) *be given where* $f \in H$ *and* H *is a minimal set in* $\mathscr{G}$ *that admits a simple zero set* $Z = M^{-1}(0)$. *Then there is an* ε_0 *such that for every* ε *with* $|\varepsilon| < \varepsilon_0$ *and for each* $f \in Z$ *there is a unique point*

$$\xi(f, \varepsilon) = \bar{u} + O(\varepsilon) \in R^2$$

such that $(\xi(f, \varepsilon), f) \in \mathscr{W}^s(\varepsilon) \cap \mathscr{W}^u(\varepsilon)$ *for* $0 < |\varepsilon| < \varepsilon_0$. *The function* ξ *is continuous. Moreover, if* $\Xi^0 = \{(\xi(f, \varepsilon), f) : f \in Z\}$ *and* $\Xi^k = \Psi^k(\Xi^0)$, *then* $\Lambda = V \cup \{\bigcup_{-\infty}^{\infty} \Xi^k\}$ *is a compact invariant set for the Poincaré map* Ψ. *For* $\varepsilon \neq 0$, *each point of* Ξ^k *is a point of transversal intersection of the stable and unstable manifolds of* V.

The same conclusions hold if H *is a compact invariant set in* $\mathscr{G}$ *that is a proper Melnikov perturbation, and* $Z \subset M^{-1}(0)$ *is a global cross section to the translational flow on* H, *where* (3.11) *is valid for every* $f \in Z$.

Proof. As described in §III.3, the separation of the stable and unstable manifold in the normal direction n is

$$d(f, \varepsilon) = \|u_s(0, f, \varepsilon) - u_u(0, f, \varepsilon)\|$$
$$= \varepsilon \nu M(f) + O(\varepsilon^2)$$

where ν is a nonzero constant. We define a coordinate system near $\bar{u}$ in R^2 as follows: Consider the map $(\alpha, \beta) \to u_s(\alpha, f, \varepsilon) + \beta n$, where n is now a unit normal vector to $F(\bar{u})$, see Figure 7. This map takes a neighborhood of the origin in R^2 onto a neighborhood of $\bar{u}$. Note that α is a coordinate along the stable manifold (essentially the time parameter) and β is a coordinate in the unit normal direction. By taking the Jacobian of this map when $\alpha = \beta = \varepsilon = 0$ and using the fact that n is normal to $F(\bar{u})$, one sees that the Jacobian is nonsingular and (α, β) constitutes a valid coordinate system near $\bar{u}$ when ε is small.

In this coordinate system the stable manifold has coordinates $(\alpha, 0)$. The unstable manifold

$$\text{(3.22)} \qquad \begin{aligned} u_u(\alpha, f, \varepsilon) &= u_s(\alpha, f, \varepsilon) \pm d(f_\alpha, \varepsilon)n \\ &= u_s(\alpha, f, \varepsilon) \pm \varepsilon \nu M(f_\alpha) + O(\varepsilon^2) \end{aligned}$$

has coordinates $(\alpha, \pm \varepsilon \nu M(f_\alpha) + O(\varepsilon^2))$.[6] In these coordinates an intersection of the stable and unstable manifold is obtained when $\varepsilon \nu M(f_\alpha) \pm O(\varepsilon^2) = 0$, that is when $G(\alpha, f, \varepsilon) = 0$ where $G(\alpha, f, \varepsilon) = M(f_\alpha) \pm O(\varepsilon)$. By assumption one has

$$G(0, f, 0) = M(f) = 0, \quad \text{and} \quad D_1 G(0, f, 0) \neq 0$$

when $f \in Z$. Thus by the Implicit Function Theorem there is an $\varepsilon_0 > 0$ such that one can find a solution $\alpha = \bar{\alpha}(f, \varepsilon)$ of $G(\alpha, f, \varepsilon) = 0$ for $f \in Z$ and $|\varepsilon| < \varepsilon_0$. In these coordinates the intersection of the stable and unstable manifold is precisely $(\alpha, \bar{\alpha}(f, \varepsilon))$, which we define to be $\xi(f, \varepsilon)$. The intersection is clearly transversal since the zeros of $M(f_t)$ are simple for $f \in Z$

Since $(\xi(f, \varepsilon), f)$ lies in both the stable and unstable bundles for V, one has

$$\Psi^k(\xi(f, \varepsilon), f) \to V \quad \text{as } k \to \pm\infty.$$

Since $(\xi(f, \varepsilon), f)$ is a typical point in Ξ^0 this shows that $\Lambda = V \cup \{\bigcup_{k=-\infty}^{\infty} \Xi^k\}$ is closed and hence compact. Clearly Λ is invariant. Since $\xi(f, \epsilon,), f) \in \Xi^0$ is a point of transversal intersection of the stable and unstable manifolds for V, and since the Poincaré mapping Ψ preserves this property, we see that each point of Ξ^k is a point of transversal intersection of the stable and unstable manifolds of V.

The argument for a compact invariant set H, which is a proper Melnikov perturbation, is identical. $\square$

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[6] The sign in (3.22) is $+$ if the inner product $\langle n, [u_u(0, f, \varepsilon) - u_s(0, f, \varepsilon)] \rangle$ is positive, otherwise the sign is minus.

Remark. A related construction of homoclinic orbits can be found in Chow, Hale and Mallet-Paret (1980).

IV. Hyperbolic sets and the Shadowing Lemma

In this section we shall concentrate on the Poincaré map considered as a discrete skew product dynamical system and develop the skew product analogs of hyperbolic sets and the Shadowing Lemma, see Bowen (1970) and Conley (1978). Also in this section the notation of the last section will be used in a slightly more general setting, since we will take the second factor to be an arbitrary discrete dynamical system. Let

$$(4.1) \qquad \Psi = (\psi, \eta) \colon R^n \times Z \to R^n \times Z \colon (x, f) \to (\psi(x, f), \eta(f))$$

define a discrete skew product dynamical system, where Z is an arbitrary compact Hausdorff space. This means that both Ψ and η are homeomorphisms. Furthermore, assume that Ψ and ψ are smooth in their first argument, i.e., $D_1 \psi(x, f)$ is defined and continuous for all $(x, f) \in R^n \times Z$. For each integer k we define $\Psi(x, f, k) \colon R^n \times Z \to R^n$ by

$$\Psi^k(x, f) = (\Psi(x, f, k), \eta^k(f)).$$

Since one has
$$\Psi(x, f, k + \ell) = \Psi(\Psi(x, f, k), \eta^k(f), \ell)$$

for all k, a simple induction argument shows that the derivative $D_1 \Psi(x, f, k)$ exists and is continuous for all $(x, f) \in R^n \times Z$.

IV.1 Skew hyperbolic sets and exponential dichotomies. The derivative operator defines a linear skew product flow (the *variational flow*) associated with the flow (4.1). In this section and the next, the variable $f \in Z$ will be considered as a parameter, but no derivatives will be taken with respect to this parameter (Note that the space Z need not be a manifold.) Let $p = (x, f)$ denote a typical point in $R^n \times Z$, k an integer, and set $q = (y, g) = \Psi^k(p)$. Define $\Phi(p, k)$ to be the linear mapping

$$\Phi(p, k)u = \Phi(x, f, k)u = D_1 \Psi(x, f, k)u,$$

so $\Phi(p, k) \colon T_p R^n \to T_q R^n$. Here we label the tangent space with the parameter $f \in Z$ also. We often identify $T_p R^n$ with R^n, conforming to the usual mathematical practice. The mapping

$$(4.2) \qquad\qquad \Pi(u, p, k) = (\Phi(p, k)u, \Psi^k(p))$$

defines a linear skew product flow on the tangent bundle of $R^n \times Z$, which we identify with $R^n \times R^n \times Z$, see Sacker and Sell (1974, 1976). Note that $\Phi(x, f, 1) = D_1 \psi(x, f)$.

Let $\Lambda \subset R^n \times Z$ be a compact invariant set for the mapping Ψ. We say that Φ admits an *exponential dichotomy* over Λ, or that Λ is a *skew hyperbolic set*

for Ψ, if there exist constants K and μ, with $0 < \mu < 1$, and a continuous mapping[7] $P: \Lambda \to \mathscr{L}(R^n, R^n)$ such that $P(p)$ is a linear projection operator which satisfies

(4.3) $$P(\Psi^k(p))\Phi(p,k) = \Phi(p,k)P(p)$$

and

(4.4)
$$\|\Phi(p,k)P(p)\| \le K\mu^k, \qquad p \in \Lambda, k \ge 0,$$
$$\|\Phi(p,k)[I - P(p)]\| \le K\mu^{-k}, \qquad p \in \Lambda, k \le 0.$$

If one defines

$$\mathscr{S}(p) = \text{ Range } P(p) \qquad \text{and} \qquad \mathscr{U}(p) = \text{ Null Space } P(p),$$

then there is a continuous splitting of the tangent bundle $T\Lambda$ given by $T_p\Lambda = \mathscr{S}(p) \oplus \mathscr{U}(p)$. Formula (4.3) says that the linear map $\Phi(p,k) = D_1\Psi(x,f,k)$ preserves this splitting, so $\Phi(p,k): \mathscr{S}(p) \to \mathscr{S}(q)$ and $\Phi(p,k): \mathscr{U}(p) \to \mathscr{U}(q)$.

Since P is continuous on Λ, this implies that for every $\varepsilon > 0$ there is an $\alpha = \alpha(\varepsilon) > 0$ such that

(4.5) $$\|P(x_1, f) - P(x_2, f)\| \le \varepsilon,$$

whenever $|x_1 - x_2| \le \alpha$ and $(x_1, f), (x_2, f) \in \Lambda$.

The following lemma will be useful for our proof. It asserts that the constant K appearing in (4.4) can be chosen to be 1.

Lemma. *Let Λ be a skew hyperbolic invariant set for Ψ. Then there is a norm $\|\cdot\|_p$ on R^n, $p \in \Lambda$, with the property that*

(4.6)
$$\begin{cases} \|\Phi(p,k)P(p)\|_q \le \mu^k, & k \ge 0, \\ \|\Phi(p,k)[I - P(p)]\|_q \le \mu^{-k}, & k \le 0, \end{cases}$$

where $q = \Psi^k(p)$.

Proof. For any $(w,p) \in R^n \times \Lambda$ we have a unique decomposition $w = u + v$ where $u \in \text{Range}(P(p))$ and $v \in \text{Null space}(P(p))$. Now define $\|w\|_p = \max\{\|u\|_p, \|v\|_p\}$ where

$$\|u\|_p = \sup\{K^{-1}\mu^k \|\Phi(p,k)u\| : k \ge 0\},$$
$$\|v\|_p = \sup\{K^{-1}\mu^k \|\Phi(p,k)v\| : k \le 0\}.$$

The verification that (4.6) holds is straightforward and we omit the details.[8]
□

In order to simplify our notation we shall drop the subscript q on the norms in (4.6). The context will make it clear which subscript is appropriate.

---

[7] This mapping is referred to as a *projector* in Sacker and Sell (1974).
[8] See Proposition 4.2 in Shub (1987).

Inequality (4.6) implies that

$$
(4.7) \qquad
\begin{aligned}
|\Phi(x,f,k)u| &\le \mu^{k}|u|, & k \ge 0, u \in \mathscr{S}(x,f), \\
|\Phi(x,f,k)v| &\ge \mu^{-k}|v|, & k \ge 0, v \in \mathscr{U}(x,f)
\end{aligned}
$$

provided that $(x,f) \in \Lambda$. Inequality (4.7) says that Φ contracts in $\mathscr{S}$ and expands in $\mathscr{U}$ as $k \to +\infty$.

Let us now return to the invariant set $\Lambda = V \cup \{\bigcup_{-\infty}^{\infty} \Xi^{k}\}$ described in the last section.

Theorem 3. *Assume that the hypotheses of Theorem 2 are satisfied, i.e., H is a minimal set in $\mathscr{G}$ that admits a simple zero set $Z = M^{-1}(0)$. Let Λ be given by Theorem 2. Let $\Psi = (\psi,\eta)$ be the Poincaré map induced on the section $S = R^{2} \times Z$ and let Φ be the variational flow given by (4.3). Then Λ is a skew hyperbolic invariant set for Ψ.*

The same conclusion is valid if H is a compact invariant set in $\mathscr{G}$ that is a proper Melnikov perturbation, and $Z \subset M^{-1}(0)$ is a global cross section to the translational flow on H where (3.11) is valid for every $f \in Z$.

Proof. This proof is somewhat standard so we will refer the reader to the literature at several points. Using the notation of §III, let $\phi(t,x_{0},f,\varepsilon)$ be the solution of (3.1) which satisfies $\phi(0,x_{0},f,\varepsilon) = x_{0}$; let $Z \subset H$ be the cross section of the translation flow defined by $Z = M^{-1}(0)$; let $T: Z \to R$ the first return time; let $\psi(x_{0},f,\varepsilon) = \phi(T(f),x_{0},f,\varepsilon)$; and let

$$\Psi = (\psi,\eta): S \times I \to S: (x_{0},f,\varepsilon) \to (\psi(x_{0},f,\varepsilon),\eta(f)),$$

where $I = (-\varepsilon_{0},\varepsilon_{0})$ and $S = R^{2} \times Z$.

By Theorem 2 of the previous section, at each point $p \in \Xi^{k}$ the stable and unstable manifolds intersect transversally; that is, if we define $E_{p}^{s} = T_{p}\mathscr{W}^{s}(\varepsilon) \cap S$ and $E_{p}^{u} = T_{p}\mathscr{W}^{u}(\varepsilon) \cap S$ as in formula (3.7), then by the definition of transversality $T_{p}R^{2} = R^{2} = E_{p}^{s} + E_{p}^{u}$ and the splitting is continuous by the smoothness of the stable and unstable manifolds. If $p \in V \cap S$, then by the same definition and the local stable manifold theorem as given for example in Hale (1969) one has $T_{p}R^{2} = E_{p}^{s} + E_{p}^{u}$.

The estimates in (4.4) follow at once from the general properties of exponential dichotomies as given in Coppel (1978) and as explicitly shown in the proof of Proposition 2.2 of Palmer (1984) for the periodic case. We note that the results given in Coppel (1978) do not require the system to be periodic and so Palmer's argument holds in the present case as well. □

IV.2 Orbits and pseudo orbits. Our next objective is to introduce the space of (skew) pseudo orbits and to study some of the dynamical properties of this space.

For $f_{0} \in Z$ we let $f = (\ldots, f_{-1}, f_{0}, f_{1}, \ldots)$ denote the η-orbit through f_{0}, i.e., f_{i} and f_{i+1} are related by $f_{i+1} = \eta(f_{i})$. The collection of all η-orbits

f will be denoted by $\mathfrak{Z}$. Thus each point f in $\mathfrak{Z}$ is an infinite bisequence. Furthermore, $\mathfrak{Z}$ is a compact invariant set under the shift flow $f \to \sigma(f)$, where $\sigma(f)_i = f_{i+1}$, and one has

$$\sigma(\ldots, f_{-1}, f_0, f_1, \ldots) = (\ldots, \eta(f_{-1}), \eta(f_0), \eta(f_1), \ldots)$$

for all $f_0 \in Z$. Next let $\mathfrak{M} = \ell_\infty$ be the Banach space of bounded bisequences

$$x = (\ldots, x_{-1}, x_0, x_1, \ldots), \qquad x_i \in R^n,$$

with the supremum norm $\|x\| = \sup_i |x_i|$. The shift flow on $\mathfrak{M}$ is given by $S(x)_i = x_{i+1}$, and the flow on $\mathfrak{M} \times \mathfrak{Z}$ is given by

$$(x, f) \to S \times \sigma(x, f) \overset{\text{def}}{=} (S(x), \sigma(f)).$$

For $(x_0, f_0) \in R^n \times Z$ we define the *orbit selection mapping*

$$(x, f) \overset{\text{def}}{=} [x_0, f_0]$$

to be the ordered pair of infinite bisequences $(x, f) \in \mathfrak{M} \times \mathfrak{Z}$ with

(4.8) $$x_{i+1} = \psi(x_i, f_i), \quad \text{for all } i.$$

A Ψ-*orbit* is defined to be an ordered pair $(x, f) \in \mathfrak{M} \times \mathfrak{Z}$ with the property that (4.8) is valid. The orbit selection mapping is the mapping of an initial condition (x_0, f_0) into the Ψ-orbit passing through (x_0, f_0). It is easily seen that if $(x, f) = [x_0, f_0]$, then

$$(S(x), \sigma(f)) = [\psi(x_0, f_0), \eta(f_0)].$$

In other words, the orbit selection mapping is a flow isomorphic mapping of $R^n \times Z$ into $\mathfrak{M} \times \mathfrak{Z}$.

Let $\Lambda \subset R^n \times Z$ be a compact invariant set for Ψ. A point $(x, f) \in \mathfrak{M} \times \mathfrak{Z}$ is said to be a (skew) α-*pseudo orbit* for $\Psi|_\Lambda$ if $(x_i, f_i) \in \Lambda$ and $|\psi(x_i, f_i) - x_{i+1}| \le \alpha$ for all i, where $\alpha > 0$. Let

$$\mathfrak{P}_\alpha \overset{\text{def}}{=} \{(x, f) \in \mathfrak{M} \times \mathfrak{Z} : (x, f) \text{ is an } \alpha\text{-pseudo orbit for } \Psi|_\Lambda \}.$$

For an α-pseudo orbit, x_{i+1} may jump (by at most α) from $\psi(x_i, f_i)$, but f is an η-orbit, i.e., no jumps in f_i. A Ψ-orbit (y, g) is said to be a β-*shadow* of an α-pseudo orbit (x, f) if $g = f$ and $|x_i - y_i| \le \beta$ for all i.

One should view $\mathfrak{M} \times \mathfrak{Z}$ as a family of Banach spaces parameterized by $f \in \mathfrak{Z}$. The tangent bundle to $\mathfrak{M} \times \mathfrak{Z}$ is $\mathfrak{E} = \mathfrak{M} \times \mathfrak{M} \times \mathfrak{Z}$. Let us now return to the skew hyperbolic invariant set $\Lambda \subset R^n \times Z$. By using the splitting of Ψ over Λ one obtains a corresponding splitting of $\mathfrak{E}$ over $\mathfrak{P}_\alpha$ in the form $\mathfrak{E}(\mathfrak{P}_\alpha) = \mathfrak{S} \oplus \mathfrak{U}$ where

$$\mathfrak{S}(x, f) = \bigotimes_{i=-\infty}^\infty \mathscr{S}(x_i, f_i), \qquad \mathfrak{U}(x, f) = \bigotimes_{i=-\infty}^\infty \mathscr{U}(x_i, f_i)$$

and $(x,f) \in \mathfrak{P}_\alpha$. What this means is that if $(x,f) \in \mathfrak{P}_\alpha$ and $w \in \mathfrak{M}$, then there is a unique decomposition $w = u + v$ with $u \in \mathfrak{S}(x,f)$ and $v \in \mathfrak{U}(x,f)$. Indeed one has

$$u_i = P(x_i,f_i)w_i, \quad \text{and} \quad v_i = [I - P(x_i,f_i)]w_i.$$

Since the projector P is uniformly bounded on the compact set Λ, this means that

$$\|\|w\|\| \overset{\text{def}}{=} \|u\| + \|v\|$$

is a norm on $\mathfrak{M}$ which is equivalent to the norm $\|w\|$. For $w \in \mathfrak{E}(x,f)$ the norm $\|\|w\|\|$ can also be written in the form

$$(4.9) \qquad \|\|w\|\| = \sup_i |P(x_i,f_i)w_i| + \sup_i |[I - P(x_i,f_i)]w_i|.$$

The equivalence of the two norms means that there is a constant $K_3 > 0$ such that

$$(4.10) \qquad K_3^{-1}\|\|w\|\| \leq \|w\| \leq K_3\|\|w\|\|.$$

IV.3 The Skew Shadowing Lemma. In this section we will prove the following result.

Theorem 4 (The Skew Shadowing Lemma). *Let Λ be a compact, skew hyperbolic invariant set for $\Psi \colon R^n \times Z \to R^n \times Z$. Then for every $\beta > 0$ there is an $\alpha > 0$ such that every α-pseudo orbit (x,f) for $\Psi|_\Lambda$ there is a β-shadow orbit (y,f). Moreover, there is a $\beta_0 > 0$ such that if $0 < \beta < \beta_0$ then the Ψ-orbit (y,f) given above is uniquely determined by the α-pseudo orbit (x,f), and the mapping $(x,f) \to (y,f)$ is continuous on $\mathfrak{P}_\alpha$.*

Proof. This is a slight extension of the proof found in Meyer and Sell (1987), and corrects an error in that proof. By modifying the functions F and f outside a compact neighborhood of the invariant set Λ, if necessary, we may assume that all orbits of Ψ are bounded.

Define a map $\mathscr{F} \colon \mathfrak{M} \times 3 \to \mathfrak{M}$ by

$$(\mathscr{F}(x,f))_i = (\psi(x_{i-1},f_{i-1}))$$

and consider f as a parameter. With f fixed, x is a fixed point of this map if and only if $x_i = \psi(x_{i-1},f_{i-1})$ for all i, i.e., (x,f) is a Ψ-orbit. Also $(x,f) \in \mathfrak{M} \times 3$ is an α-pseudo orbit if and only if $\|\mathscr{F}(x,f) - x\| \leq \alpha$. The Ψ-orbit (y,g) is a β-shadow for an α-pseudo orbit (x,f) if and only if $\mathscr{F}(y,f) = y$, $g = f$, and $\|x - y\| \leq \beta$. (Recall that the β-shadow and the α-pseudo orbit must have the same η-orbit f).

The main conclusion of the Skew Shadowing Lemma can then be reformulated to read: For every $\beta > 0$ there is an $\alpha > 0$ such that if $\|\mathscr{F}(x,f) - x\| \leq \alpha$, then $\mathscr{F}$ has a fixed point y in an β-neighborhood of x. This formulation suggest the use of the Inverse Function Theorem, which we will describe later.

For simplicity define the function $\mathscr{G}(x,f) = \mathscr{F}(x,f) - x$. We then seek a zero of $\mathscr{G}$ for fixed f. The function $\mathscr{G}$ is continuously differentiable with respect to its first argument with the derivative given by $D_1\mathscr{G}(x,f)$ where

(4.11)
$$(D_1\mathscr{G}(x,f)w)_i = D_1\psi(x_{i-1},f_{i-1})w_{i-1} - w_i = \Phi(x_{i-1},f_{i-1},1)w_{i-1} - w_i.$$

We claim that there are constants $\alpha_0 > 0$ and K_0 such that

$$(4.12) \qquad \|D_1\mathscr{G}(x,f)\| \le K_0 \quad \text{and} \quad \|D_1\mathscr{G}(x,f)^{-1}\| \le K_0$$

for all $(x,f) \in \mathfrak{P}_{\alpha_0}$.

The first estimate in (4.12) is elementary since $D_1\psi$ is uniformly bounded on the compact set Λ. In order to verify the second estimate, it suffices to show that $D_1\mathscr{G}(x,f)$ has a uniform lower bound for $(x,f) \in \mathfrak{P}_{\alpha_0}$, for some $\alpha_0 > 0$. We will verify this lower bound in terms of the equivalent norm $\||w\|| = \|u\| + \|v\|$.

The first step is to show that there is an $a > 0$ such that

$$(4.13) \qquad \begin{aligned} a\|u\| &\le \|D_1\mathscr{G}(x,f)u\|, & u \in \mathfrak{S}(x,f), \\ a\|v\| &\le \|D_1\mathscr{G}(x,f)v\|, & v \in \mathfrak{U}(x,f), \end{aligned}$$

provided that $(x,f) \in \mathfrak{P}_\alpha, \alpha > 0$. Recall that $\|u\| = \sup_i|u_i|$. Using (4.7) and (4.11) with $(x,f) \in \mathfrak{P}_\alpha$ and $u \in \mathscr{S}(x,f)$ we get

$$\begin{aligned} |(D_1\mathscr{G}(x,f)u)_i| &= |\Phi(x_{i-1},f_{i-1},1)u_{i-1} - u_i| \ge |u_i| - |\Phi(x_{i-1},f_{i-1},1)u_{i-1}| \\ &\ge |u_i| - \mu|u_{i-1}| \ge |u_i| - \mu\|u\|. \end{aligned}$$

Hence $|(D_1\mathscr{G}(x,f)u)_i| + \mu\|u\| \ge |u_i|$, which implies that $\|D_1\mathscr{G}(x,f)u\| \ge (1-\mu)\|u\|$. Similarly if $v \in \mathfrak{U}(x,f)$ one has

$$\begin{aligned} |(D_1\mathscr{G}(x,f)v)_i| &= |\Phi(x_{i-1},f_{i-1},1)v_{i-1} - v_i| \ge |\Phi(x_{i-1},f_{i-1},1)v_{i-1}| - |v_i| \\ &\ge \mu^{-1}|v_{i-1}| - |v_i| \ge \mu^{-1}|v_{i-1}| - \|v\|. \end{aligned}$$

Hence $|(D_1\mathscr{G}(x,f)v)_i| + \|v\| \ge \mu^{-1}|v_{i-1}|$, which implies that $\|D_1\mathscr{G}(x,f)v\| \ge (\mu^{-1} - 1)\|v\|$ and completes the proof of (4.13).

The next step is to show that there is a constant K_1 such that for every $\varepsilon > 0$ there is an $\alpha > 0$ satisfying

$$(4.14) \qquad \|D_1\mathscr{G}(x,f)u\| + \|D_1\mathscr{G}(x,f)v\| \le \||D_1\mathscr{G}(x,f)w\|| + K_1\varepsilon\||w\||$$

for all $(x,f) \in \mathfrak{P}_\alpha$. For $\varepsilon > 0$ let $\alpha = \alpha(\varepsilon) > 0$ be chosen so that (4.5) is valid. If $(x,f) \in \mathfrak{P}_\alpha$, then $|\psi(x_{i-1},f_{i-1}) - x_i| \le \alpha$ for all i. Consequently one has

$$(4.15) \qquad \|P(\Psi(x_{i-1},f_{i-1})) - P(x_i,f_i)\| \le \varepsilon, \quad \text{for all } i.$$

Let $(x,f) \in \mathfrak{P}_\alpha$ and set $w = u + v \in \mathfrak{M}$, where $u \in \mathfrak{S}(x,f)$ and $v \in \mathfrak{U}(x,f)$. Then one has $P(x_i,f_i)w_i = u_i$ and $[I - P(x_i,f_i)]w_i = v_i$. Furthermore from

(4.8) and (4.11) one has

$$
\begin{aligned}
(D_1 \mathscr{G}(x,f)u)_i &= \Phi(x_{i-1}, f_{i-1}, 1)u_{i-1} - u_i \\
&= \Phi(x_{i-1}, f_{i-1}, 1)P(x_{i-1}, f_{i-1})w_{i-1} - P(x_i, f_i)w_i \\
&= P(\Psi(x_{i-1}, f_{i-1}))\Phi(x_{i-1}, f_{i-1}, 1)w_{i-1} - P(x_i, f_i)w_i \\
&= P(x_i, f_i)[\Phi(x_{i-1}, f_{i-1}, 1)w_{i-1} - w_i] \\
&\quad + [P(\Psi(x_{i-1}, f_{i-1})) - P(x_i, f_i)]\Phi(x_{i-1}, f_{i-1}, 1)w_{i-1} \\
&= P(x_i, f_i)(D_1 \mathscr{G}(x,f)w)_i \\
&\quad + [P(\Psi(x_{i-1}, f_{i-1})) - P(x_i, f_i)]\Phi(x_{i-1}, f_{i-1}, 1)w_{i-1}.
\end{aligned}
$$

Consequently from (4.13) one has

$$
\|D_1 \mathscr{G}(x,f)u\| \le \sup_i |P(x_i, f_i)(D_1 \mathscr{G}(x,f)w)_i| + K_2 \varepsilon \|w\|,
$$

where $K_2 = \sup\{\|\Phi(x_0, f_0, 1)\| : (x_0, f_0) \in \Lambda\}$. Similarly one obtains

$$
\|D_1 \mathscr{G}(x,f)v\| \le \sup_i |[I - P(x_i, f_i)](D_1 \mathscr{G}(x,f)w)_i| + K_2 \varepsilon \|w\|.
$$

By adding the last two inequalities and using (4.9) and (4.10), we obtain (4.14) with $K_1 = 2K_2 K_3$. Inequalities (4.13) and (4.14) imply that

$$
a\|\|w\|\| \le \|\|D_1 \mathscr{G}(x,f)w\|\| + K_1 \varepsilon \|\|w\|\|.
$$

By fixing ε so that $0 < K_1 \varepsilon < a$ and setting $\alpha_0 = \alpha(\varepsilon)$ one obtains

$$
(a - K_1 \varepsilon)\|\|w\|\| \le \|\|D_1 \mathscr{G}(x,f)w\|\|, \qquad (x,f) \in \mathfrak{P}_{\alpha_0},
$$

which completes the proof of (4.12).

Because of the continuity of $D_1 \mathscr{G}$ and $D_1 \mathscr{G}^{-1}$, inequality (4.12) can be extended to hold in some neighborhood of $\mathfrak{P}_{\alpha_0}$. In particular, there is a $\delta_0 > 0$ and a K_1 such that

(4.16) $\|D_1 \mathscr{G}(x,f)\| \le K_1$ and $\|D_1 \mathscr{G}(x,f)^{-1}\| \le K_1$

for all $(x,f) \in \mathfrak{M} \times \mathfrak{Z}$ with the property that $\operatorname{dist}((x,f), \mathfrak{P}_{\alpha_0}) \le \delta_0$.

The following version of the Inverse Function Theorem with estimate can be found in Hartman (1964).

Inverse Function Theorem. *Let X be a Banach space, $B_\delta(x^0)$ the ball of radius δ about $x^0 \in X$, and $G: B_\delta(x^0) \to X$ a C^1 function with $y^0 = G(x^0)$. Assume that $DG(x)$ has a bounded inverse with*

$$
\|DG(x)\| \le K \quad and \quad \|DG^{-1}(x)\| \le K
$$

for all $x \in B_\delta(x^0)$, where K is a constant. Let $\rho = \delta/K^2$ and $\sigma = \delta/K$, then there exists a domain Ω, with $B_\rho(x^0) \subset \Omega \subset B_\delta(x^0)$ and such that G is one-to-one on Ω. Moreover, $B_\sigma(y^0) \subset G(\Omega)$. In particular, for every $y \in B_\sigma(y^0)$ there

is a unique $x \in \Omega$ *with* $y = G(x)$, *and the mapping* $x = G^{-1}(y)$ *is continuous on* $B_\sigma(y^0)$.

Apply this version of the inverse function theorem to the function $G(x) = \mathscr{G}(x, f)$ and consider $f \in Z$ as a parameter. For $\beta > 0$ we define

$$\alpha = \min\left(\frac{\beta}{2K_1}, \alpha_0\right), \qquad \delta = \min(\beta, \delta_0).$$

Let (x, f) be any fixed α-pseudo orbit and set $y^0 = G(x)$. Then $\|y^0\| = \|G(x)\| \le \alpha$. If $\hat{x} \in B_\delta(x)$, then $\|\hat{x} - x\| < \delta \le \delta_0$. Hence (4.16) holds for all $(\hat{x}, f)$ where $\hat{x} \in B_\delta(x)$. Let $\sigma = \delta/K_1$ be given by the Inverse Function Theorem. Then $\sigma = 2\alpha$, and consequently one has $0 \in B_\sigma(y^0)$. By the Inverse Function Theorem there is a $y \in B_\delta(x)$ with $G(y) = 0$, i.e., (y, f) is an Ψ-orbit. Furthermore, $\|y - x\| < \delta \le \beta$, so (y, f) is a β-shadow of (x, f).

If we take $\beta_0 = \delta_0/K_1^2$ then the inverse function theorem yields that G is one-to-one on $B_{\beta_0}(x)$ for $x \in \mathfrak{P}_{\alpha_0}$. Thus the distance between zeros of G is at least β_0 and this proves the uniqueness part of the Shadowing Lemma. $\square$

Remark. By the uniformity of the estimates given above the Ψ-orbits constructed here have a form of distality. That is, if (x, f) and $(\overline{x}, f)$ are Ψ-orbits constructed above with $x \neq \overline{x}$ then $\|x - \overline{x}\| > \beta_0$ so for some i,

$$\|x_i - \overline{x}_i\| = \|\psi^i(x_0, f_0) - \psi^i(\overline{x}_0, f_0)\| > \beta_0.$$

V. The Shadowing Lemma establishes Bernoulli bundles

Henceforth, assume that H is a compact invariant set in $\mathscr{G}$ and that H is a proper Melnikov perturbation with global cross section $Z \subset M^{-1}(0) \subset H$, where every $f \in Z$ satisfies (3.11). Thus for small ε, $\Lambda = V \cup \{\bigcup_{-\infty}^{\infty} \Xi^k\}$, as defined in Theorems 2 and 3, is a skew hyperbolic invariant set for the Poincaré map $\Psi = (\psi, \eta)$. Let K be the transition matrix given by (2.7) and $A : B_n(K) \to B_n(K)$ the subshift of finite type which is defined by K. (The size n of K will be given within the proof of Theorem 5 below.) Let $A \times \eta : B_n(K) \times Z \to B_n(K) \times Z$ be the product map defined by (2.8) and let $A \otimes \eta$ denote the induced flow on the Bernoulli bundle $B_n(K) \otimes Z$.

Theorem 5. *Let the above assumptions be satisfied. There is an* $\varepsilon_0 > 0$ *such that for every* ε *with* $0 < |\varepsilon| < \varepsilon_0$ *there is an integer* n_0 *such that for all* $n \ge n_0$, *there is a compact invariant set* $\Omega_n \subset R^2 \times Z$ *for the Poincaré map* Ψ *such that the restriction* $\Psi|_{\Omega_n}$ *is equivalent to the product map* $A \times \eta$.

Proof. Assume that $\varepsilon_0 > 0$ is so small that the conclusions of Theorems 2 and 3 hold for $0 < |\varepsilon| < \varepsilon_0$. Hence $\Lambda = V \cup \{\bigcup_{-\infty}^{\infty} \Xi^k\}$ is a compact skew hyperbolic set for the Poincaré map $\Psi : R^2 \times Z \to R^2 \times Z$. Let $\beta \le \beta_0$ be as given in the uniqueness part of the Shadowing Lemma. We assume β is small enough that $\text{dist}(\Xi^0, \Xi^i) > 4\beta$ for $i \neq 0$; and let α be determined by

the Skew Shadowing Lemma. Further restrict ε_0 so that by Theorem 0 for all $0 < |\varepsilon| < \varepsilon_0$ the invariant set V is within $\alpha/4$ of $\{0\} \times Z$. Since $\Xi^k \to V$ as $k \to \pm\infty$ there is a $\kappa > 0$ such that $\text{dist}\,(\Xi^k, V) < \alpha/4$ when $|k| \geq \kappa$. Let N be the $\frac{\alpha}{2}$-neighborhood of V.
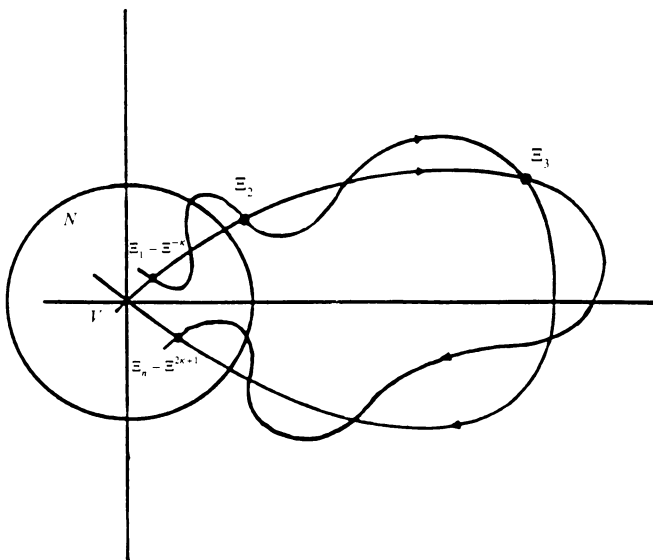

FIGURE 9. A portion of the hyperbolic set Λ

In order to simplify the proof somewhat we will relabel some of the Ξ^k with subscripts as follows: As stated above $\Xi^{-\kappa}$ lies inside the neighborhood N. Since it will represent our starting point let it be denoted by Ξ_1 and, in general, let $\Xi_i = \Psi^{i-1}(\Xi_1) = \Xi^{-\kappa+i-1}$ for $1 < i \leq n_0$ where $n_0 = 2\kappa + 1$. Also by the above for all $n \geq n_0$, Ξ_n is in N. The Poincaré map takes the set Ξ_1, which is in N, to the set Ξ_2, then to the set Ξ_3, etc. until it taken to Ξ_n, which is again back in N. See Figure 9. [9] (In the periodic case the Poincaré map is just a diffeomorphism of the plane and the sets Ξ_i are just points as illustrated in Figure 9.) By the choice of the neighborhood N of V any jump within N is allowed in an α-pseudo orbit and in particular remaining at Ξ_n is allowed.

Let $n \geq n_0$ be fixed. We will now describe the α-pseudo orbits for (3.1) which are of particular interest here. As always an α-pseudo orbit is a bisequence (x, f) with $f_{i+1} = \eta(f_i)$ and $|\psi(x_i, f_i) - x_{i+i}| \leq \alpha$ for all i, however we require more. In addition we ask that for each i there is an r with $1 \leq r \leq n$ and such that $(x_i, f_i) \in \Xi_r$. Furthermore we require that if $(x_i, f_i) \in \Xi_r$ then one can have $(x_{i+1}, f_{i+1}) \in \Xi_s$ only if $k_{rs} = 1$, where k_{rs} is the rs-term in the transition matrix K given by (2.7). Finally with the exception of the case where $(x_i, f_i) \in \Xi_n$ we require that $x_{i+1} = \psi(x_i, f_i)$.

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[9] Compare Figures 5 and 9.

We wish to write a code for these α-pseudo orbits. In particular we shall parameterize them by q where $q \in B_n(K)$, the Bernoulli space. For each $f \in Z$ and $1 \leq \ell \leq n$, where $n \geq n_0$, there is a unique point $\xi_\ell(f, \varepsilon)$ in R^2 with $(\xi_\ell(f, \varepsilon), f) \in \Xi_\ell$ by Theorem 2. Let $q \in B_n(K)$ and let $z_i = \eta^i(f)$ be an η-orbit. Then $\{(\xi_{q_i}(z_i, \varepsilon), z_i)\}$ is an α-pseudo orbit. By the Skew Shadow Lemma there is a point $p \in R^2$ such that $\{\Psi^i(p, f)\}$ is β-shadow for this α-pseudo orbit. Define a map $\Upsilon : B_n(K) \times Z \to R^2 \times Z$ by $\Upsilon(q, f) = (p, f)$ where p is defined by the process given above. Υ is continuous by the Skew Shadowing Lemma.

To see that Υ is one-to-one let $\Upsilon(q, f) = (p, f)$ and $\Upsilon(\bar{q}, f) = (\bar{p}, f)$ where $q \neq \bar{q}$. The Υ-orbits through (p, f) and $(\bar{p}, f)$ are β-shadows of different pseudo-orbits and, since dist$(\Xi^0, \Xi^i) > 4\beta$ for $i \neq 0$, these pseudo-orbits differ by at least 4β in one entry, say the kth. Since the actual orbits β-shadow these pseudo orbits dist$(\Psi^k(p, f), \Psi^k(\bar{p}, f)) > 2\beta$ and these Ψ-orbits are different. Thus Υ is a continuous, one-to-one mapping of a compact Hausdorff space, and so it is a homeomorphism onto its image $\Omega_n \subset R^2 \times Z$.

Let $(q, f) \in B_n(K) \times Z$, $(r, g) = (A(q), \eta(f))$, $\Upsilon(q, f) = (p, f)$, $\Upsilon(r, g) = (s, g)$. By the above construction the Ψ-orbits through $\Psi(p, f)$ and (s, g) are β-shadows of each other. By uniqueness $\Psi^{i+1}(p, f) = \Psi^i(s, g)$ or

$$\Psi \circ T(q, f) = \Upsilon \circ (A \times \eta)(q, f).$$

Consequently $\Psi|_{\Omega_n}$ is equivalent to $A \times \eta$. In other words, the diagram

$$
\begin{array}{ccc}
B_n(K) \times Z & \xrightarrow{\;A \times \eta\;} & B_n(K) \times Z \\
\Upsilon \downarrow & & \downarrow \Upsilon \\
\Omega_n & \xrightarrow{\;\Psi\;} & \Omega_n
\end{array}
$$

is commutative. $\square$

The set Ω_n is an invariant set for the Poincaré map Ψ. This generates an invariant set $[\Omega_n]$ in the skew product flow generated by (3.1). As a matter of fact $([\Omega_n], \Pi)$ is the suspension of the discrete flow (Ω_n, Ψ) and Ω_n is a global cross section for $[\Omega_n]$. Since $(B_n(K) \times Z, A^* \eta)$ and $(\Omega_n, A \otimes \eta)$ are flow isomorphic, it follows that the continuous flows $(\mathscr{B}_n(K), \Psi)$ and $([\Omega_n], \Pi)$ are flow isomorphic. We have thus proved the following result:

Theorem 6. *Let H be a compact invariant set in $\mathscr{G}$ which is a proper Melnikov perturbation. Then there is an $\varepsilon_0 > 0$ such that for every ε with $0 < |\varepsilon| < \varepsilon_0$ there is an integer n_0 such that for every $n \geq n_0$ the skew product flow generated by (3.1) has an invariant set $[\Omega_n]$ which is flow isomorphic to the flow on the Bernoulli bundle $\mathscr{B}_n(K) = B_n(K) \otimes H$.*

VI. AN ILLUSTRATIVE EXAMPLE

We return to Duffing's equation but now with a different perturbation. In particular consider the coupled 2-parameter equation

$$\dot{x}_1 = x_2,$$

(6.1) $$\dot{x}_2 = x_1 - x_1^3 + \varepsilon_1 f(x, \theta, \varepsilon),$$

$$\dot{\theta} = \omega + \varepsilon_2 h(\theta) + \varepsilon_1 g(x, \theta, \varepsilon)$$

on $R^2 \times T^2$ where f, g and h are smooth, $x = (x_1, x_2) \in R^2$, $\theta = (\theta_1, \theta_2) \in T^2$, $\omega = (\omega_1, \omega_2)$ is constant, and $\varepsilon = (\varepsilon_1, \varepsilon_2)$ is the bifurcation parameter. To be more specific assume that

(6.2) $$f(x, \theta, 0) = a_1 \cos \theta_1 + a_2 \cos \theta_2 \stackrel{\text{def}}{=} E(\theta),$$

where a_1, a_2 are nonzero constants. (Note that $E(\theta)$ does not depend on x.)

Let $x^\varepsilon(t) = x^\varepsilon(t, x_0, \theta_0)$ and $\theta^\varepsilon(t) = \theta^\varepsilon(t, x_0, \theta_0)$ denote the solution of (6.1) with initial condition $x^\varepsilon(0) = x_0$ and $\theta^\varepsilon(0) = \theta_0$. At $\varepsilon_1 = 0, x^{(0, \varepsilon_2)}(t) \equiv 0$ is a hyperbolic fixed point for (6.1). Furthermore there is homoclinic orbit for (6.1) whenever $\varepsilon_1 = 0$. If $\varepsilon_2 \neq 0$, then the solution of the θ-equation is generally not almost periodic. One expects to find locking-in when $\varepsilon_2 \neq 0$.

Even though the system (6.1) is given in a special form, i.e., the function h is independent of x, we believe that the properties described in the last paragraph are generic among 2-parameter perturbations of Duffing's equation which are close to a quasi periodic perturbation, see Section 11.3 in Chow and Hale (1982). In particular, for generic 2-parameter perturbations, with perturbation parameter $\varepsilon = (\varepsilon_1, \varepsilon_2)$, and which reduce to

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 - x_1^3, \quad \dot{\theta} = \omega$$

at $\varepsilon = 0$, one expects to be find a 1-dimensional curve Γ in the parameter space with $0 \in \Gamma$ and such that the perturbed system has a homoclinic orbit for $\varepsilon \in \Gamma$. One should view (6.2) as an added simplifying assumption which is used for illustrative purposes only.

As shown in §III.5, the Melnikov transform $M(E(\theta^0(\cdot)))$ has a simple zero set $Z^0 \subset H = T^2$ at $\varepsilon = 0$, for a large open set W in the parameter space $\{(a_1, a_2, \omega_1, \omega_2)\}$. Furthermore Z^0 is also a global cross section for the flow θ^0 on T^2. Let $(a_1, a_2, \omega_1, \omega_2)$ be a fixed element of W.

Since $\theta^{(0, \varepsilon_2)}(t) \rightarrow \theta^{(0,0)}(t)$ as $\varepsilon_2 \rightarrow 0$, uniformly for t in compact sets in R, Z^0 is a global cross section for the flow $\theta^{(0, \varepsilon_2)}$ on T^2 for ε_2 small and nonzero. If $\theta_0 \in Z^0$, then $M(E(\theta^0(\tau + \cdot)))$ has a simple zero at some τ. By continuity, for ε_2 small the perturbed Melnikov function has a simple zero at $\hat{\theta}_0$ near θ. The collection $Z^{(0, \varepsilon_2)}$ of all such $\hat{\theta}_0$ forms another global cross section to the flow $\theta^{(0, \varepsilon_2)}$ on T^2.

In this way the system (6.1) reduces to a 1-parameter family (the parameter being ε_2) where the hypotheses of Theorem 5 are satisfied. Note that for $\varepsilon_2 \neq 0$

the flow $\theta^{(0,\varepsilon_2)}$ on T^2 is generally not almost periodic. Because of Theorem 5, we see that there is an $\varepsilon_0 > 0$ such that for $0 < |\varepsilon_1| < \varepsilon_0$ and $0 \leq |\varepsilon_2| < \varepsilon_0$, the full system (6.1) contains a Bernoulli bundle flow $[\Omega_n] \sim B^n(K) \otimes T^2$. The bundle $[\Omega_n]$, as well as the induced flow on $[\Omega_n]$, varies continuously in the various parameters: $a_1, a_2, \omega_1, \omega_2, \varepsilon_1$ and ε_2.

In the case where ω_1/ω_2 is rational, (6.1) reduces to a periodic perturbation of Duffing's equation. By using the classical Melnikov construction, which applies in this case, one can prove the existence of a Bernoulli shift in (6.1). However even in this setting, our methods give more information. The Bernoulli bundle $\mathscr{B}_n(K) = B_n(K) \otimes T^2$ which we construct is a bundle over the torus T^2. The classical Melnikov construction would lead only to a bundle $B_n(K) \otimes S^1$ over the circle S^1.

An addendum. After this paper was completed we learned of the related work of Daniel M. Stoffer, *Some geometric and numerical methods for perturbed integrable systems*, Dissertation, ETH, Zürich, 1987. Also see the new book S. Wiggins, *Global bifurcations and chaos*, Appl. Math. Sci., No. 73, Springer-Verlag, New York, 1988.

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